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NUMERICAL TRIGONOMETRY

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NUMERICAL TRIGONOMETRY

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PREFACE

IT seems to be generally agreed that it is desirable to cut out a considerable amount of heavy manipulation from the elementary arithmetic and algebra course and to use the time thus saved in giving the boy some acquaintance with numerical Trigonometry. [See Report on Teaching of Elementary Algebra and Numerical Trigonometry issued by the Mathematical Association Committee, 1911.] To test a boy's patience, accuracy, and ability to produce an orderly piece of work, it is not necessary to make him plough through pages of complicated exercises in Vulgar Fractions. Scope can be found for all these qualities in numerical problems connected with geometry, physics, mechanics, etc., and the range of such problems is greatly increased if he has an elementary knowledge of Trigonometry.

The work involved in solving triangles is often complicated, and must be done carefully and in an orderly manner. There is a certain amount of drudgery involved, but it is drudgery with some interest attached to it.

Young boys with no particular mathematical bent are found to take very kindly to this numerical Trigonometry, more kindly indeed than to other parts of their elementary mathematics.

This book is an attempt to indicate the kind of work that can profitably be done, and the contents are well within the powers of boys of from 13 to 15 years of age. It is adapted from the first part of my *Trigonometry for Beginners.* Much of it is the same, but there has been some re-arrangement, some parts have been re-written, and certain excisions and additions have been made.

The ratios are introduced gradually and their meanings driven home by easy problems. The tangent is taken first since it is used in the easiest kind of height and distance problem, and it is well if it can be introduced in connection with some actual problem met with in field-work.

Four-figure tables are used from the outset.

The Sine and Cosine appear next in problems involving the hypotenuse, and a set of miscellaneous problems shows that even with this slight equipment much can be done. The Secant, Cosecant, and Cotangent then appear as the reciprocals of the Cosine, Sine, and Tangent; but these may be omitted, although their use sometimes simplifies matters. The general triangle is now solved by splitting into right-angled triangles. The arithmetic up to this point is designedly kept easy and only the natural ratios are used. The next stage is the solution of right-angled and other triangles in cases where the arithmetic is heavier and logarithms are used to lighten it. The methods are the same as before—the only difference is in the way in which the arithmetical operations are carried out. After a little time, log. sines, etc. are brought in to effect a still further saving of labour.

The next chapter consists of Exercises in generalizing, leading up to the Sine and Cosine formulae and their use in solution of triangles. For those who wish to omit the

Sine and Cosine formulae, Chapters I—V and Exs. XLI, followed by Chapters VI and VII and Exs. XLII will be enough. Others may prefer to take Chapters VIII and IX with the worked examples not involving logarithms and Exs. XLI after Chapter V.

A short chapter on Degree of Approximation will be found at the end; each case is dealt with on its own merits and no attempt made to formulate any general rules.

It is assumed that everyone who uses the book has some collection of four-figure tables.

I have made the Examples as varied as possible, and have tried to emphasize throughout the importance of orderly arrangement of work, and of checking of results.

J. W. M.

April, 1912.

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CHAPTER I

THE TANGENT OF AN ACUTE ANGLE

1. As the name implies, the main purpose of Trigonometry is measurement of triangles (Greek *τρίγωνον*, a triangle, and *μέτρια*, measurement). Every triangle has six "parts," viz. three sides and three angles, and, as we shall see, if we are given a certain number of these the rest can be calculated. For this purpose we make use of what are called "trigonometrical ratios" of angles. We shall eventually become acquainted with six of these, but for the present we shall use only three, the Sine, Cosine, and Tangent.

2. Definitions. Angles of elevation and depression.

If A, B be two points at different heights and a horizontal line AX be drawn through A in the vertical plane containing A and B, the angle between AX and AB is called the **angle of elevation** of B from A if A is below B and the **angle of depression** of B from A if A is above B.

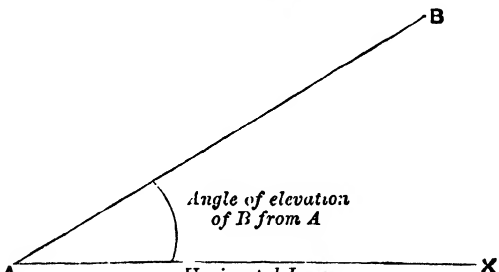


Fig. 1.

Or, Suppose an observer stationed at A, facing B, and holding to his eye the hinge of a pair of dividers. If he keeps one leg of the dividers pointing horizontally, and swings the

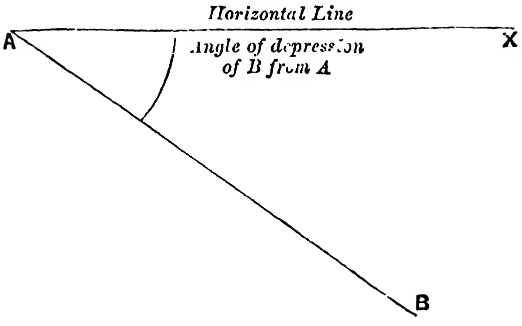


Fig. 2.

other leg up or down, as the case may be, until it points to B, the angle through which the dividers are opened is the angle of elevation or depression of B from A.

3. *Caution.* Suppose a man at A on the top of a tower AB observing an object C below. The angle of depression of C from A is *not* BAC.

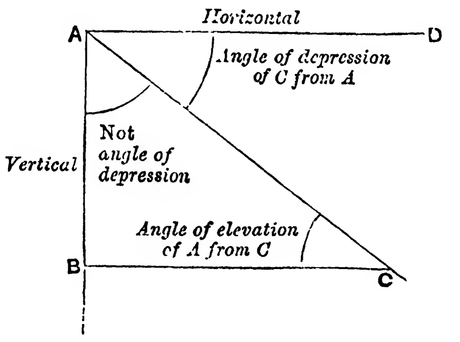


Fig. 3.

We must draw a horizontal line AD through A, i.e. a line perpendicular to AB. Then DAC is the angle.

It is obvious from Fig. 3 that the angle of depression of C from A = the angle of elevation of A from C.

4. Consider the following problem which the class may have met in doing field work: "The angle of elevation of the top of a tree at a distance of 80 feet in the horizontal plane through the foot of the tree is 35°; find the height of the tree." Let the class solve this problem by drawing, each choosing his own scale. All should obtain the same result, 56 feet, although the actual

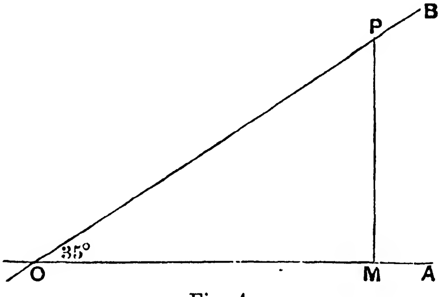


Fig. 4.

lines measured may be of different lengths. This simply means that in all right-angled triangles such as OMP (Fig. 4) in which MOP = 35°, MP is always the same fraction (about ·7) of OM.

5. To emphasise this point, draw an angle of 35°, AOB. Take any point P in one arm OB and draw PM perpendicular to OA.

Measure MP and OM and work out $\frac{MP}{OM}$ to two decimal places. Do this for several positions of P.

[In class work let each member of the class take P at random and work out $\frac{MP}{OM}$ separately. The same plan will be useful in other cases.]

Collect the results and arrange them in tabular form:

MP ins.	OM ins.	$\frac{MP}{OM}$
1·91	2·73	·70
......	......	...

MP and OM will be different every time, but $\frac{MP}{OM}$ should always be the same.

Do the same thing, taking P in the other arm OA and drawing PM perpendicular to OB.

6. *Definition.* This ratio $\frac{MP}{OM}$ is called the tangent of the angle AOB.

Notation. The result obtained in § 5 is written tan 35° = ·70.

7. The tangent of any acute angle can be found graphically in the same way. OM is in the denominator, and since the actual length of OM is immaterial, it is convenient to draw the

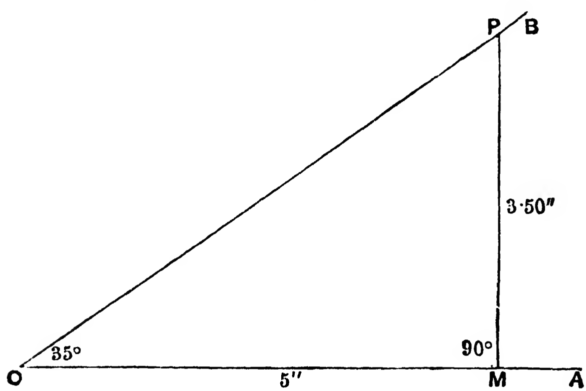


Fig. 5.

figure so that OM contains an exact number of inches or centimetres; the division involved in finding $\frac{MP}{OM}$ will then be much simplified.

For instance, suppose we want to find tan 35°.

Draw an angle of 35°, AOB (Fig. 5). Along OA take OM containing an exact number of inches, say 5, and draw MP perpendicular to OA meeting OB in P. Measure MP (3·50 inches).

Then $$\tan 35° = \frac{MP}{OM} = \frac{3{\cdot}50}{5} = {\cdot}70.$$

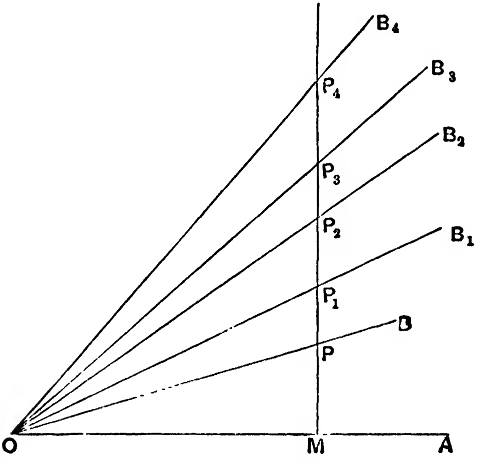


Fig. 6.

8. It is evident (Fig. 6) that as we change ∠ AOB, we change also the shape of the triangle PMO. If we always keep OM the same length, MP will increase or decrease with ∠ AOB. Thus we shall get a different tangent for every angle; in other words, every acute angle has its own tangent which belongs to it and to no other.

EXERCISES. I.

1. Use squared paper to find the tangents of

$$10°, 20°, 30°, 40°, 50°, 60°, 70°, 80°.$$

[Draw large figures. A slight error made in measuring your lines will affect your result less seriously than an error of equal amount made in a small figure.]

2. In finding $\tan 35°$ as in § 5, suppose that MP and OM are found by measurement to be 1·91 and 2·73 inches, and that each of these may be in error by ·01 inch. Show that all we can say with certainty about $\tan 35°$ is that it lies between ·693 and ·706.

9. The table of natural tangents. The values of the tangent have been calculated by methods of higher mathematics, and are given in ordinary collections of four-figure tables for all acute angles at intervals of 6′ or ·1°, and by means of the difference columns we can find the values for angles at intervals of 1′. The values given are correct to four decimal places. Thus $\tan 30°$ is given as ·5774. This means that it is between ·57735 and ·57745.

EXERCISES. II.

1. Check, by means of the tables, your results in Exs. I.

2. Write down the tangents of

(i) 15°, 23°, 37°, 56°, 72°, 85°;
(ii) 9° 18′, 27° 36′, 38° 54′, 53° 12′, 62° 24′, 85° 48′;
(iii) 11·2°, 31·5°, 47·4°, 54·7°, 73·9°, 81·3°;
(iv) 11° 11′, 26° 50′, 40° 28′, 55° 40, 63° 10′, 78° 49′.

10. Another way of defining the tangent of an acute angle.

Let ABC (Fig. 7) be a right-angled triangle in which C is the right angle. Then of the two sides which contain the right angle, BC is *opposite* to A and AC is *adjacent* to A.

$$\tan A = \frac{BC}{AC}\left(\frac{\text{side opposite to A}}{\text{side adjacent to A}}\right),$$

$$\tan B = \frac{AC}{BC}\left(\frac{\text{side opposite to B}}{\text{side adjacent to B}}\right),$$

or $$BC = AC \times \tan A \quad\text{and}\quad AC = BC \times \tan B.$$

Thus tan A may be defined as a multiplier. If any right-angled triangle be drawn containing an acute angle A, tan A is the multiplier which converts the side adjacent to A into the side opposite to A.

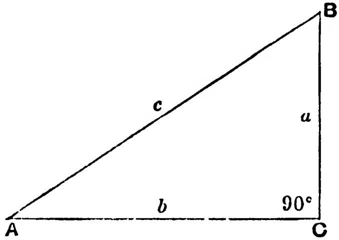


Fig. 7.

We might therefore have solved our problem of § 4 without drawing a figure to scale. For if F, T be the foot and top of the tree and O the position of the observer, TFO is a right-angled triangle with $\angle$ FOT = 35°. The table tells us that

$$\tan 35° = \cdot 7001,$$

$$\text{i.e. } FT = OF \times \cdot 7001.$$

$$\therefore \text{ Height of tree} = 80 \times \cdot 7001 \text{ ft.} = 56 \text{ ft. nearly.}$$

EXERCISES. III.

1. A tower stands on a horizontal plane. A man standing on the plane observes the angle of elevation of the top of the tower. Find the height of the tower if

(i) Distance of man from tower = 100 ft., angle of elevation = 30°;
(ii) .... = 50 ft., ... = 60°;
(iii) .... = 200 ft., ... = 20°;
(iv) .... = 80 ft., ... = 70°.

2. Find the distance of the man from the tower if

(i) Height of tower = 100 ft., angle of elevation = 40°;
(ii) .... = 30 ft., ... = 30°;
(iii) .... = 320 ft., ... = 45°;
(iv) .... = 120 ft., ... = 65°.

3. A man lies on a cliff and observes the angle of depression of a boat. Find the distance of the boat from the foot of the cliff if

(i) Height of cliff $=300$ ft., angle of depression $=20°$;
(ii) .... $=250$ ft., ... $=25°$;
(iii) .... $=400$ ft., ... $=35°$.

4. A man whose eye is 5 ft. 6 ins. above the ground stands 80 ft. away from a tower and finds the angle of elevation of the top to be 64°. Find the height of the tower above the ground.

5. At what distance from a tower 200 ft. high must a man 6 ft. high stand, so that the angle of elevation of the top may be 60°?

11. *Notation.* In a triangle ABC it is usual to denote the angles by capital letters A, B, C, and the number of units of length in the sides opposite to them by small letters a, b, c. Thus a is opposite to A, etc. (Fig. 7).

12. *Example.* In $\triangle$ABC, $C = 90°$, $A = 60°$, $b = 10$, find a.

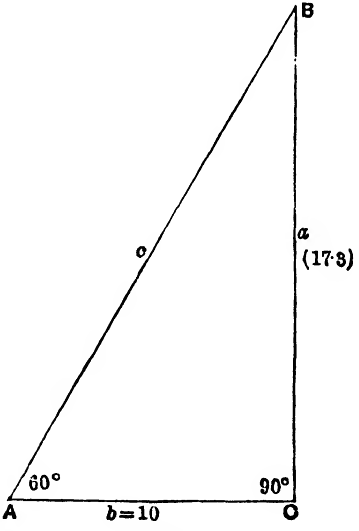


Fig. 8.

The multiplier which converts b into a is tan A,

$$\text{or} \quad a = b \tan A.$$

$$\therefore \quad a = 10 \times \tan 60^\circ$$
$$= 10 \times 1{\cdot}7321$$
$$= 17{\cdot}321.$$

We might have said $b = a \tan B$,

$$\therefore 10 = a \times {\cdot}5774,$$

$$\therefore \quad a = \frac{10}{{\cdot}5774}$$
$$= 17{\cdot}32.$$

The process of multiplying 10 by 1·7321 being easier than that of dividing 10 by ·5774, we see that if b is given and a is wanted, it is better to use the multiplier which converts b into a than that which converts a into b. In other words it is better to fill up the blank in $a = b \times ?$ than in $b = a \times ?$.

Check the solution by drawing to scale.

13. Notice that as the table only gives the tangent correct to four decimal places, our results will not as a rule be correct to four decimal places.

E.g. suppose we want $125 \tan 60^\circ$ and we multiply 125 by 1·7321; we get 216·5125. But all the table tells us is that $\tan 60^\circ$ is between 1·73205 and 1·73215, i.e. 1·7321 may be ·00005 in error, $\therefore$ 216·5125 may be $125 \times {\cdot}00005$ or ·00625 in error, or $125 \tan 60^\circ = 216{\cdot}5125 \pm {\cdot}00625$, i.e. it is between 216·50625 and 216·51875.

EXERCISES. IV.

1. In $\triangle$ABC, C = 90°. Find a, given

(i) A = 35°, b = 100; (iv) A = 87°, b = 2000;
(ii) A = 45°, b = 73; (v) B = 51°, b = 20;
(iii) A = 65°, b = 300; (vi) B = 73°, b = 120.

2. In $\triangle$ABC, C = 90°. Find b, given

(i) A = 35°, a = 100; (iv) B = 78°, a = 40;
(ii) A = 45°, a = 73; (v) A = 50° 12′, a = 10;
(iii) A = 15°, a = 300; (vi) B = 61° 48′, a = 50.

3. ABC is an isosceles triangle, having AB = AC.

AD is drawn perpendicular to BC.

If BC = 8 cms., and $\angle$ABC = 55°, find AD.

[Notice that AD divides the isosceles triangle into two congruent right-angled triangles. This should be remembered in all questions on isosceles triangles.]

4. ABC is an isosceles triangle, having AB = AC.

AD is drawn perpendicular to BC.

If AD = 10 cms., and $\angle$ABC = 42°, find BC.

5. The side of an equilateral triangle is 2″. Find the altitude.

6. The altitude of an equilateral triangle is 2″. Find the side.

7. The acute angle of a rhombus is 40° and the shorter diagonal is 8 cms. Find the other diagonal.

8. The obtuse angle of a rhombus is 130° and the shorter diagonal is 6 cms. Find the other diagonal.

9. One angle of a rhombus is double the other. The longer diagonal is 10 ins. Find the other diagonal.

10. ABCD is a square, side 10 cms. AC is joined and $\angle$BAC bisected by a line meeting BC in E. Calculate the lengths of BE and EC.

11. In an isosceles triangle, the base is 6″ and vertical angle 70°. Find the other angles and the height.

12. In an isosceles triangle the altitude is 10″ and each of the base angles 50°. Find the base.

13. The diagonal of a rectangle makes an angle of 35° with the longer side. If the shorter side is 8″, find the length of the longer side.

14. In the triangle ABC, the altitude AD is 10″, B = 40°, C = 60°. Find BC. [Find BD and DC.]

15. In the triangle ABC, the altitude AD is 10″, B = 20°, C = 110°. Find BC.

Definition. Two angles whose sum is a right angle are called **complementary** angles. Each is called the **complement** of the other.

16. Write down the complements of

35°, 43°, 71°, 28° 19′, 43° 43′, 64° 28′.

[Notice that if two angles are complementary, a right-angled triangle can be constructed containing both of them.]

17. Draw a right-angled triangle having $C=90°$, $AC=3''$, $BC=4''$.

What is $\tan A$? What is $\tan B$? What is $\tan A \times \tan B$?

18. Draw any right-angled triangle with acute angles A and B. Let the lengths of the sides opposite to A and B be a'' and b''.

What is $\tan A$? What is $\tan B$? What is $\tan A \times \tan B$?

19. What is the relation between the tangents of complementary angles?

Verify this by working out

(i) $\tan 30° \times \tan 60°$; (ii) $\tan 41° \times \tan 49°$;

(iii) $\tan 16° 15' \times \tan 73° 45'$.

Account for the slight difference between what you get and what you expected to get.

20. [To be done by drawing.] ABC is a right-angled triangle having $C=90°$ and $A=60°$, $AC=10$ cms. Find CB.

If AD, AE divide A into three equal parts, what are the lengths of CD, CE?

21. From the last example, which is greater

(i) $\tan 40°$ or $2 \tan 20°$?

(ii) $\tan 60°$ or $3 \tan 20°$?

(iii) $\tan 60°$ or $\tan 40° + \tan 20°$?

22. [To be done by drawing.] ABC is a right-angled triangle having $C=90°$ and $A=75°$, AD, AE are drawn to BC so that $CAD=60°$ and $CAE=15°$. Which is greater, BD or EC?

From the figure, what are $\tan 75°$, $\tan 60°$, $\tan 15°$?

Which is greater, $\tan 75°$ or $\tan 60° + \tan 15°$?

Verify from tables.

23. From tables, find which is greater

(i) $\tan 48°$ or $2 \tan 24°$;

(ii) $\tan 80°$ or $\tan 55° + \tan 25°$;

(iii) $\tan 89°$ or $\tan 20° + \tan 30° + \tan 39°$.

Definition. Suppose an observer stationed at A, and let the North-South and East-West lines be drawn through A. The **bearing** of an object B from A tells us the direction in which the line AB points. It gives us the angle through which a man looking N. or S. would have to turn towards E. or W. in order to look along AB.

In Fig. 9, the bearings from A of B_1, B_2, B_3, B_4 are N. 60° E., N. 17° W., S. 20° W., S. 55° E. respectively.

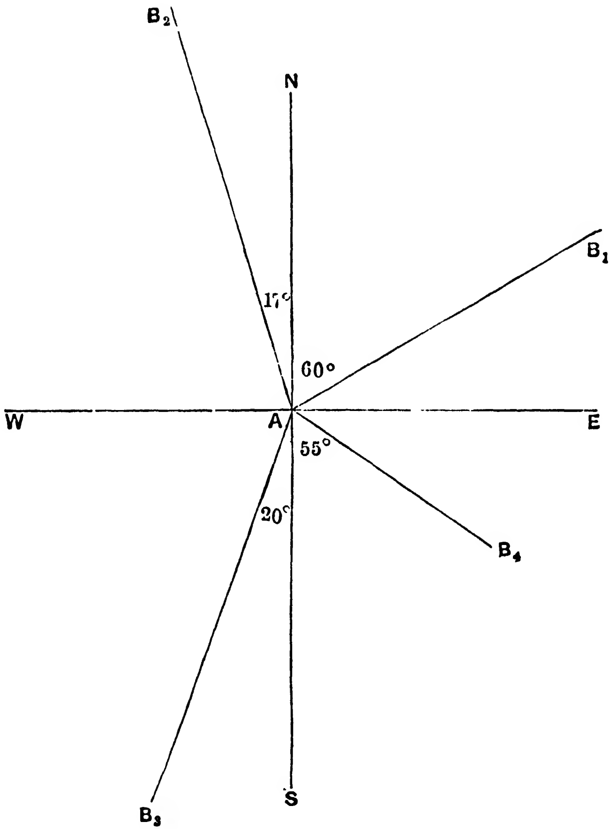


Fig. 9.

24. B is 15 miles S. of A. C bears due W. from B and S. 38° W. from A. Find the distance of B from C.

25. From a ship, A bears N. 55° W. When she has sailed 10 miles due W., A is due N. How far is she from A at the second observation?

26. A road ABCD consists of three portions, AB running due E., BC due S. and CD due E. BC is 100 yards long. A church tower (T) lies in CB produced. From a point in CD 100 yards from C the bearing of T is N. 25° W. How far is T from B, and what is the bearing of T from a point in AB 100 yards from B?

27. A ladder stands against a wall and makes an angle of 63° with the ground. If the foot is 20 feet from the wall, what is the height of the top of the ladder?

28. A ladder makes an angle of 58° with the ground and reaches a window 40 ft. high. How far is the foot of the ladder from the wall?

29. A man at the top of a cliff sees two boats lying so that the line joining them points to the foot of the cliff. Find the distance between the boats, given:—height of cliff = 200 ft., angles of depression of the boats 20° and 40°.

30. A tower is surmounted by a flagstaff. From a point on the ground 80 ft. from the foot of the tower, the angle of elevation of the top of the tower is 51° and of the top of the flagstaff 57°. Find the length of the flagstaff.

31. From points on opposite sides of a tree 50 ft. high two men observe its angles of elevation to be 25° and 35°. How far are they apart?

14. Since every different acute angle has its own tangent which belongs to it and to no other, we ought to be able to identify an angle if we are given its tangent. This is equivalent to saying that if in a right-angled triangle we are given the ratio of the sides containing the right angle, its shape is determined and its angles are fixed.

15. Construct any number of right-angled triangles having $C = 90°$ and $a = 2b$. In each case measure the angles A and B. All the results should be the same [see note to § 5].

From your figures what is the angle whose tangent is 2?

What is the angle whose tangent is $\frac{1}{2}$?

16. We see from this that in order to construct the acute angle whose tangent is 2, we want a triangle ABC with C a right angle and $\frac{BC}{AC} = 2$. Then A (opposite to BC) is the required angle.

17. *Another Example.* Construct the angle whose tangent is ·36.

Here we want a triangle ABC with C a right angle and $\frac{BC}{AC} = \cdot 36$ or $BC = AC \times \cdot 36$. Choose any convenient length for AC, say 5″. Draw CB perpendicular to AC of length $5 \times \cdot 36$ or 1·80″. Join AB. Then BAC is the required angle for

$$\tan BAC = \frac{1 \cdot 80}{5} = \cdot 36 \text{ (Fig. 10).}$$

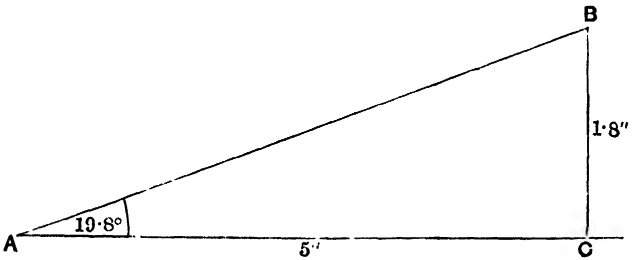


Fig. 10.

EXERCISES. V.

1. Find by drawing the angles whose tangents are

$$\frac{3}{2},\quad \frac{2}{3},\quad \cdot 48,\quad \cdot 8,\quad 3 \cdot 14,\quad \frac{9}{7}.$$

2. What ought to be the relation between the first two angles in Ex. **1**? Is this relation satisfied by your angles?

3. From the tables find the angles whose tangents are

(i) ·2493, ·7002, 4·3315, 1·1501, ·1051, 2·2460;

(ii) ·3979, 1·9883, 2·6889, ·9293, 7·0264, ·2401;

(iii) ·7146, 1·1729, 4, ·2, ·0125, ·4383;

(iv) 2·0413, 3·0061, 5, 2·035, $\frac{61}{12}$, $\frac{1}{11}$.

4. Check by tables your answers to Ex. **1**.

5. Verify, by using tables, that $A+B=90°$, if

(i) $\tan A=\frac{4}{5}$, $\tan B=\frac{5}{4}$;

(ii) $\tan A=\frac{7}{12}$, $\tan B=\frac{12}{7}$;

(iii) $\tan A=\frac{8}{15}$, $\tan B=\frac{15}{8}$.

6. In the triangle ABC, $C=90°$, $a=3$, $b=4$. What is $\tan A$? What is $\tan B$? What is A? What is B? Verify that $A+B=90°$.

7. In the triangle ABC, $C=90°$. Find A and B, given

(i) $a=7$, $b=10$;	(iv) $b=35$, $a=20$;
(ii) $a=87$, $b=100$;	(v) $b=68$, $a=51$;
(iii) $a=3271$, $b=1000$;	(vi) $b=336$, $a=2500$.

8. In an isosceles triangle the base is 6″ and the altitude 10″. Find the angles.

9. The sides of a rectangle are 7″ and 10″. Find the angle the diagonal makes with the longer side.

10. The diagonals of a rhombus are 16 and 20 cms. Find the angles.

11. A ship sails 20 miles due E. and then 10 miles due N. What is its bearing from the starting point?

12. A ship sails 25 miles due N. and then 35 miles due W. What is its bearing from the starting point?

13. Calculate the acute angle between the x-axis and the line joining the points, if the unit length is the same along both axes.

(i) (1, 2), (2, 3); (ii) (2, 3), (−2, 5); (iii) (2, −3), (−3, 2); (iv) (3, 4), (−1, −3); (v) (0, 0), (5, 1).

14. What will be the angle of elevation of the top of a tower to an observer if

(i) Height of tower = 100 ft., his distance from foot = 100 ft.?
(ii) .... = 100 ft., ... = 50 ft.?
(iii) .... = 150 ft., ... = 200 ft.?

15. What will be the angle of depression of a boat to an observer on a cliff if

(i) Height of cliff = 330 ft., distance of boat from foot = 1 mile?
(ii) .... = 400 ft., ... = 800 yds.?
(iii) .... = 80 yds., ... = 2000 ft.?

16. At a distance of 50 ft. from the foot of a tower the angle of elevation of the top is 65°. What will it be at a distance of 100 ft.?

CHAPTER II

THE SINE AND COSINE OF AN ACUTE ANGLE

18. Suppose we try to use the tangent table in the solution of a problem involving the hypotenuse of a right-angled triangle, for instance:

$$A = 90^\circ, \quad B = 35^\circ, \quad a = 8, \text{ find } b.$$

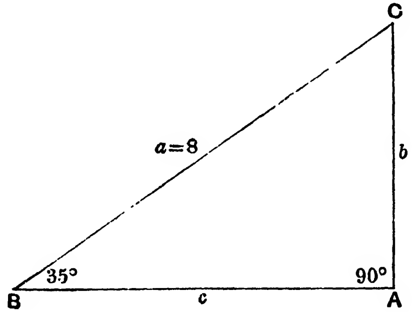


Fig. 11.

We shall have to do something like this:

$$c = b \tan 55^\circ = b \times 1{\cdot}4281$$

and
$$b^2 + c^2 = a^2,$$

$$\therefore\ b^2 (1 + 2{\cdot}039) = 64,$$

$$\therefore\ b = \sqrt{\frac{64}{3{\cdot}039}} = \frac{8}{1{\cdot}744} = 8 \times {\cdot}5734 = 4{\cdot}587.$$

[The tables of squares, square roots and reciprocals are useful here.]

This is a clumsy process and could be avoided if we were able at once to express b in terms of a, i.e. if we could at once write down the multiplier which converts a into b.

Now in all right-angled triangles such as OMP (Fig. 4) it is just as clear that MP is always the same fraction of OP as it is that MP is always the same fraction of OM.

19. To emphasise this point, draw an angle of 35°, AOB. Take any point P in one arm, OB, and draw PM perpendicular to OA. Measure OP, MP, OM, and work out the fractions $\frac{MP}{OP}$, $\frac{OM}{OP}$ each to two decimal places [see note on § 5].

Tabulate results.

OP ins.	MP ins.	OM ins.	$\frac{MP}{OP}$	$\frac{OM}{OP}$
3·82	2·19	3·13	·57	·82
......	......	......	...	...

OP, MP, OM will be different every time, but $\frac{MP}{OP}$, $\frac{OM}{OP}$ should always be the same.

Do the same thing, taking P in OA and drawing PM perpendicular to OB.

20. *Definition.* The ratio $\frac{MP}{OP}$ is called the sine of $\angle$ AOB,

.... $\frac{OM}{OP}$ .... cosine.........

Notation. The results obtained in § 19 are written

$$\sin 35^\circ = \cdot 57, \quad \cos 35^\circ = \cdot 82.$$

21. The sine and cosine of any acute angle may be found graphically in the same way. OP occurs in the denominator, and since its actual length is immaterial it is convenient to make OP contain an exact number of cms. or ins.

E.g. to find sin 35°. Draw an angle of 35°, AOB (Fig. 12). Along OB measure OP, 5″ long. Draw PM perpendicular to OA. Measure MP (2·87″) and OM (4·10″).

Then
$$\sin 35^\circ = \frac{MP}{OP} = \frac{2{\cdot}87}{5} = {\cdot}57,$$
$$\cos 35^\circ = \frac{OM}{OP} = \frac{4{\cdot}10}{5} = {\cdot}82.$$

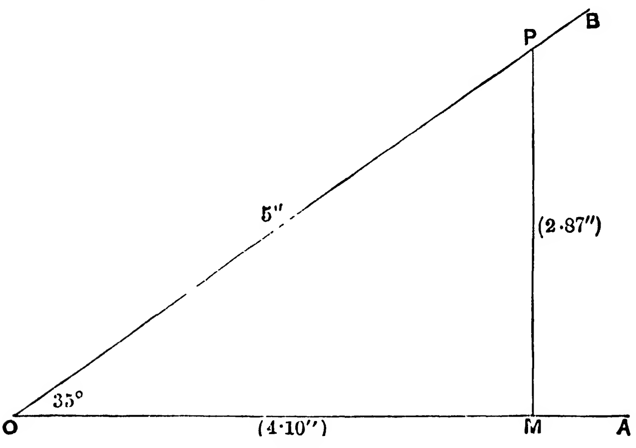


Fig. 12.

22. Draw a quadrant of a circle, OAB, and a number of radii, OP, OQ, OR, etc. (Fig. 13).

Draw Pp, Qq, Rr, etc. perpendicular to OA.

We have a number of right-angled triangles OpP, OqQ, etc. with equal hypotenuses OP, OQ, etc.

What are the sines of AOP, AOQ, AOR?

.................cosines.....................?

Does the sine of an acute angle increase or decrease as the angle increases?

Does the cosine of an acute angle increase or decrease as the angle increases?

It is clear from this figure that every different acute angle has its own sine and cosine which belong to it and to no other.

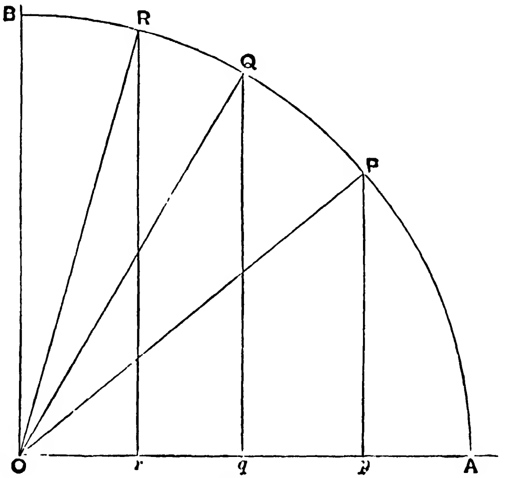


Fig. 13.

EXERCISES. VI.

1. Find by drawing the sines and cosines of

$$10°,\ 20°,\ 30°,\ 40°,\ 50°,\ 60°,\ 70°,\ 80°.$$

2. If the lengths of OP, MP, OM given in § 19 may each be in error by ·01″, what may we say with certainty about sin 35° and cos 35°? [See Exs. I. 2.]

3. In Fig. 7, § 10, what are sin A, cos A, sin B, cos B?

If two angles are complementary what can you say about their sines and cosines?

23. The tables of Natural Sines and Cosines.

The values of the sine and cosine for all acute angles at intervals of 1′ are given in the tables of Natural Sines and Cosines [see § 9]. Since the sine of an angle is equal to the cosine of its complement [Exs. VI. 3], it is possible to include the two tables in one, and this is done in some collections.

EXERCISES. VII.

1. Check by means of the tables the results of Exs. VI. 1.

2. Write down the sines of

$$18°,\ 31°,\ 40°,\ 50°,\ 62°,\ 72°;$$
$$9°\,18',\ 27°\,36',\ 38°\,54',\ 51°\,6',\ \ 62°\,24',\ 85°\,21';$$
$$11°\,11',\ 26°\,50',\ 40°\,28',\ 55°\,40',\ 63°\,10',\ 78°\,49'.$$

3. Write down the cosines of the same angles.

4. Having done Ex. **2**, which of the answers to Ex. **3** could you have written down without referring to the tables?

24. Another way of defining the sine and cosine of an acute angle.

Let ABC be a right-angled triangle in which C is the right angle (Fig. 7, § 10).

Then
$$\sin A = \frac{CB}{AB}\left(\frac{\text{side opposite to } A}{\text{hypotenuse}}\right),$$
$$\cos A = \frac{AC}{AB}\left(\frac{\text{side adjacent to } A}{\text{hypotenuse}}\right);$$

i.e. $CB = AB \times \sin A$ and $AC = AB \times \cos A$.

Similarly $AC = AB \times \sin B$ and $CB = AB \times \cos B$.

Thus sin A and cos A may be defined as multipliers. If any right-angled triangle be drawn containing an acute angle A, sin A is the multiplier which converts the hypotenuse into the side opposite to A and cos A is the multiplier which converts the hypotenuse into the side adjacent to A.

Thus in the problem proposed at the beginning of this chapter, the multiplier which converts a into b is sin 35° which by the tables is ·5736,

$$\therefore\ b = 8 \times \cdot 5736 = 4\cdot 589.$$

EXERCISES. VIII.

1. In $\triangle$ABC, C = 90°. Find a and b, given

(i)	$c=10$,	A = 35°;	(iv)	$c=70$,	B = 28°;
(ii)	$c=8$,	B = 46°;	(v)	$c=5$,	B = 81°;
(iii)	$c=100$,	A = 74°;	(vi)	$c=12$,	A = 52°.

2. Draw a right-angled triangle having BC 5 inches long, a right angle at A and an angle of 37° at B. Draw AD perpendicular to BC. On your figure mark the size of every angle. There are three right-angled triangles ABC, DBA, DAC. Measure AB, AC, AD, BD, CD and calculate from each triangle the sine, cosine, and tangent of 37° and 53°.

3. Construct a triangle ABC having BC 10 cms., CA 9 cms., AB 7 cms. Draw the perpendiculars of the triangle and measure them. Then find the sines of the angles of the triangle each in two ways. Now measure the angles with a protractor and find their sines from the tables.

4. In an isosceles triangle the vertical angle is 70°, and each of the equal sides is 8″. Find the other angles and the base.

5. A ladder, 30 feet long, stands against a vertical wall. If it makes an angle of 70° with the ground, what is the height above the ground of the point at which it touches the wall? What is the distance of the foot of the ladder from the wall?

6. If the ladder is pulled away from the wall until it makes an angle of 60° with the ground, how far is each end of the ladder from its first position?

7. C is 6 miles N. 61° E. from A. B is E. of A and S. of C. Find the distances of B from A and C.

8. A is N. of B and E. of C. C is 10 miles N. 55° W. from B. Find AC and AB.

9. C bears N. 38° E. from B. A is N. of B, and N. 52° W. of C. If A is 8 miles from B, how far is C from A and B?

10. A is N. 48° W. of C. B is N. 42° E. from C and E. of A. If AB = 5 miles, how far is A from C?

11. The side of a rhombus is 10 cms. and its acute angle 70°. Find the lengths of the diagonals.

12. AB is a chord of a circle, centre O, radius 8 cms. The angle AOB is 110°. Find the length of AB.

13. The legs of a pair of dividers are 5″ long. Find the distance between the points if the dividers are opened to an angle of 54°.

14. A man walks 1000 yards up a slope of 5°. How high is he above the horizontal plane through his starting point?

15. A ship sails due E. and then due N., and reaches a spot 20 miles N. 75° E. from the starting point. How far did the ship sail N. and how far E.?

16. [To simplify the arithmetic, take

$$\left.\begin{matrix}\sin 30^\circ\\ \cos 60^\circ\end{matrix}\right\} = \frac{1}{2}, \quad \left.\begin{matrix}\sin 36^\circ 52'\\ \cos 53^\circ\ 8'\end{matrix}\right\} = \frac{3}{5}, \quad \left.\begin{matrix}\sin 41^\circ 48'\\ \cos 48^\circ 12'\end{matrix}\right\} = \frac{2}{3},$$

$$\left.\begin{matrix}\sin 48^\circ 36'\\ \cos 41^\circ 24'\end{matrix}\right\} = \frac{3}{4}, \quad \left.\begin{matrix}\sin 53^\circ\ 8'\\ \cos 36^\circ 52'\end{matrix}\right\} = \frac{4}{5}.$$

Look these out in the tables and see that the values are very near to those given.]

In $\triangle$ABC:

(i) Given A = 90°, C = 30°, c = 12, find a;
(ii) Given C = 90°, A = 36° 52′, a = 9, find c;
(iii) Given B = 90°, C = 41° 48′, c = 12, find b;
(iv) Given A = 90°, C = 53° 8′, b = 6, find a;
(v) Given C = 90°, A = 41° 24′, b = 12, find c;
(vi) Given B = 90°, A = 48° 12′, c = 10, find b;
(vii) Given A = 90°, B = 60°, c = 8, find a;
(viii) Given A = 90°, C = 56°, b = 27, find a;
(ix) Given B = 90°, C = 41°, a = 9, find b;
(x) Given C = 90°, B = 72°, b = 14, find c.

25. Since every different acute angle has its own sine and cosine which belong to it and to no other, we ought to be able to identify an angle if we are given its sine or cosine.

This is equivalent to saying that if in a right-angled triangle we are given the ratio of one of the other sides to the hypotenuse, its angles are fixed.

26. Construct any number of right-angled triangles in each of which one of the short sides, BC, is $\frac{1}{3}$ of the hypotenuse AB. Measure the angle A. All the results should be the same.

From your figures, what is the angle whose sine is $\frac{1}{3}$? What is the angle whose cosine is $\frac{1}{3}$? Look out in the tables the angle whose sine is $\frac{1}{3}$ (·3333) and the angle whose cosine is $\frac{1}{3}$.

27. We see from this that in order to construct the angle whose sine is $\frac{1}{3}$, we must make a right-angled triangle ABC with right angle C, so that $\frac{BC}{AB} = \frac{1}{3}$. Then A will be the required angle.

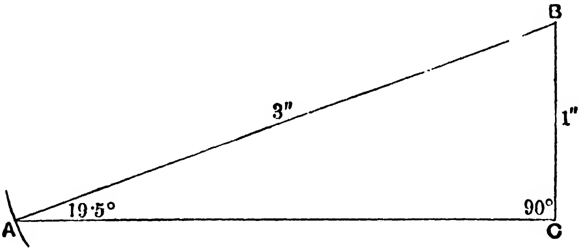


Fig. 14.

There are two ways of doing this:

(i) Choose any convenient length for AB, say 3″. Then BC must be 1″. Draw BC 1″ long. Through C draw CA perpendicular to CB of indefinite length and with centre B and radius 3″ describe a circle cutting CA in A. Join AB. Then CAB is the required angle for

$$\sin CAB = \frac{CB}{AB} = \frac{1}{3}.$$

(ii) Draw AB 3″ long. On it as diameter describe a semicircle. With centre B and radius 1″ describe a circle cutting the circumference of the semicircle in C. Join AC. Then BCA, being an angle in a semicircle, is a right angle;

$$\therefore \sin \text{BAC} = \frac{\text{CB}}{\text{AB}} = \frac{1}{3}.$$

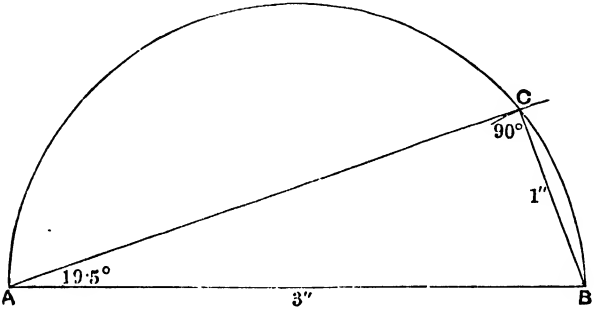


Fig. 15.

EXERCISES. IX.

1. Construct the angle whose sine is ·36 (two ways).

2. Construct the angle whose cosine is ·36.

3. What is the relation between the angles of Exs. 1 and 2?

4. Construct the angles whose sines are $\frac{2}{3}$, ·45, ·84, $\frac{3}{5}$.

5. Construct the angles whose cosines are $\frac{2}{3}$, ·64, ·8, $\frac{1}{4}$.

6. From the tables find the angles whose sines are

   (i) ·2756, ·4067, ·6947, ·7193, ·8480, ·9613;
   (ii) ·4115, ·5948, ·7108, ·9191, ·9478, ·2181.

7. From the tables find the angles whose cosines are

   (i) ·9272, ·2079, ·5878, ·6947, ·3090, ·2756;
   (ii) ·7108, ·3355, ·7826, ·9823, ·5948, ·0906.

8. From the tables find the angles whose sines are

$$\cdot 3835,\ \cdot 7005,\ \cdot 9261,\ \cdot 8025,\ \cdot 4200,\ \cdot 6218;$$

and the angles whose cosines are

$$\cdot 4000,\ \cdot 6218,\ \cdot 7\dot{6}01,\ \cdot 3835,\ \cdot 5683,\ \cdot 926.$$

9. Check by tables the results of Exs. **1–5**.

10. In $\triangle$ABC, $C=90°$, $a=3$, $c=4$. What is sin A? What is A?

11. In $\triangle$ABC, $C=90°$, $b=2$, $c=5$. What is cos A? What is A?

12. In $\triangle$ABC, $C=90°$. Find A and B, given

(i) $a=3,\ c=5$;	(iii) $b=53,\ c=100$;	(v) $a=48,\ c=60$;
(ii) $a=7,\ c=10$;	(iv) $b=1,\ c=2$;	(vi) $b=118,\ c=500$.

13. In a triangle the sides are 6″, 10″, 10″. Find the angles.

14. A ladder 20 feet long is placed against a wall, so that it reaches a point on the wall 14 feet high. Find the inclination of the ladder to the vertical.

15. A is 6 miles W. of B. C is N. of B and 10 miles from A. Find the bearing of C from A.

16. For every mile a man walks up a hill he rises 88 yards. What is the angle of slope of the hill?

17. The legs of a pair of dividers are 12 cms. long. They are opened so that the distance between the points is 7 cms. Find the angle between the legs.

18. Find the angle subtended at the centre of a circle of 5″ radius by a chord 4″ long.

19. Two tangents are drawn to a circle of 1″ radius from a point 3″ from the centre. Find the angle between them.

20. A trap-door 2 ft. 6 ins. by 2 ft. in the horizontal floor of a room is opened to an angle of 30° the hinge being in a long side. What angle does a diagonal make with the horizontal?

28. *Definition.* If from the ends of a finite straight line AB, perpendiculars AP, BQ are drawn to a straight line XY of unlimited length, PQ is called the *projection* of AB on XY (Fig. 16).

EXERCISES. X.

1. B is 10 miles N. 32° E. from A. What is the length of the projection of AB (i) on a line running E., (ii) on a line running N.?

2. In a triangle ABC, $B=60°$, $C=40°$, $BC=5''$. What are the projections of BC on the other sides of the triangle?

3. Do the same, given $B=120°$, $C=40°$, $BC=5''$.

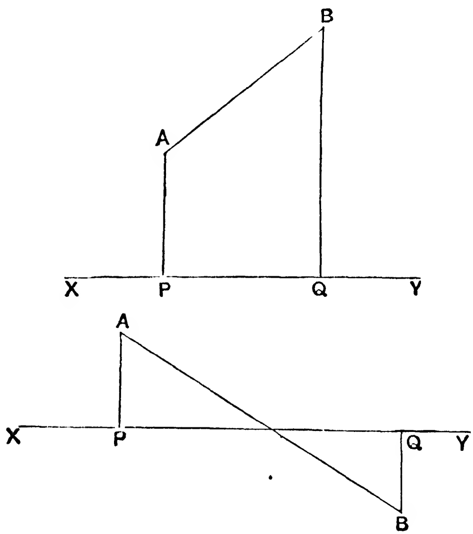


Fig. 16.

4. C is any point in the line XY. CA and CB are drawn on the same side of XY so that $CA=4''$, $CB=5''$, $\angle XCA=38°$, $\angle YCB=58°$. Find the projection of AB on XY.

5. Do the same when $\angle YCA=38°$, $\angle YCB=58°$.

6. B is 3 miles N. of A; C is 4 miles N. 50° E. from B; D is 5 miles E. from C; what is the projection of AD (i) on a line running E., (ii) on a line running N.?

7. [N.B. In a map two points A and B are represented by their projections on a horizontal plane, so that if A and B are on a hill-side the distance between them is greater than that shown on the map.]

AB is inclined at 8° to the horizon. If AB = 1000 ft., what will be the distance given by the map?

8. The projection of AB (8″ long) on a line XY is 5″. What is the acute angle between AB and XY?

9. ABC is an acute-angled triangle, in which AB = 5″. The projections of AB and AC on BC are 4″ and 3″ respectively. Find BC and the angles of the triangle.

10. The projections of AB on lines running N. and E. are respectively 5 and 8 inches. Find the length and direction of AB.

11. B is 4 miles N. 30° E. from A. C is 5 miles N. 50° E. from B. What are the projections on a line running N. of AB, BC, AC?
.... E. .... ?

12. In Ex. **11** find the distance and bearing of C from A.

13. B is 4 miles N. 30° E. from A. C is 5 miles N. 50° W. from B. Find the distance and bearing of C from A.

14. A force of 7 lbs. wt. acts in a direction making an angle of 35° with a line OX. What are its components in directions parallel and perpendicular to OX?

15. Forces 5 and 8 lbs. wt. act at a point in directions N. 20° E. and N. 40° E. respectively. Find the magnitude and direction of the resultant.

Miscellaneous Exercises on Sine, Cosine, and Tangent.

EXERCISES. XI.

1. In △ABC, C = 90°, $a=7$, $b=24$. What is c? Write down sin A, cos A, tan A, sin B, cos B, tan B. [Answers as fractions.]

2. AOB, COD are two lines intersecting at right angles. Write down the sine, cosine, and tangent of each of the angles ACO, BCO, CBO, DBO, BDO, ADO, DAO, CAO. $\left[\text{E.g. } \sin \text{ACO} = \frac{\text{AO}}{\text{AC}}.\right]$

3. If in the last Example OA = 15 cms., OB = 48 cms., OC = 20 cms., OD = 36 cms., find the lengths of AC, CB, BD, DA and the size of the angles ACB, CBD, BDA, DAC.

4. Construct the angle whose tangent is ·36, and from the figure find its sine and cosine. [You may draw any other lines you think fit.]

Check from the tables, by looking out the angle whose tangent is ·36 and then the sine and cosine of this angle.

5. Construct the angle whose sine is ·55, and find its cosine and tangent. Check from tables.

6. [N.B. $(\sin A)^2$ is written $\sin^2 A$.]
In Ex. **1** what is $\sin^2 A+\cos^2 A$?

7. From tables verify that $\sin^2 A+\cos^2 A=1$ (very nearly), when $A=$ (i) 20°; (ii) 36°; (iii) 63°.

8. Prove $\sin^2 A+\cos^2 A=1$ for any acute angle. [Use a right-angled triangle whose hypotenuse contains 1 unit of length.]
Why do you not get 1 exactly in Ex. **7**?

9. Verify $\frac{\sin A}{\cos A}=\tan A$ when $A=$(i) 25°; (ii) 48°; (iii) 60°.

10. Prove $\frac{\sin A}{\cos A}=\tan A$, for any acute angle.

11. Given $\sin A=\frac{8}{17}$, calculate $\cos A$ and $\tan A$.

[Draw a right-angled triangle ABC, having the right angle C, $c=17$, $a=8$. Then calculate b.]

12. Given $\sin A=\frac{8}{17}$, use Ex. **8** to calculate $\cos A$ and then Ex. **10** to calculate $\tan A$.

13. Convert $\frac{8}{17}$ into a decimal to four places. Hence from tables find the angle whose sine is $\frac{8}{17}$ and look out cosine and tangent of this angle.

Note. If we are given one of the three ratios, $\sin A$, $\cos A$, and $\tan A$, Exs. **11**, **12**, **13** give three ways of calculating the other two.

14. By all three methods find

(i) $\sin A$, $\cos A$, given $\tan A=\frac{3}{4}$;

(ii) $\sin A$, $\tan A$, given $\cos A=\frac{5}{13}$;

(iii) $\cos A$, $\tan A$, given $\sin A=\frac{40}{41}$.

15. Draw an equilateral triangle ABC, side 2 inches. What is the size of $\angle$A? Draw CD perpendicular to AB. What is the size of $\angle$ACD? What is the length of AD? Use Pythagoras' theorem to calculate CD. From the figure find sine, cosine, and tangent of 30° and 60°.
Compare with tables.

16. Draw a right-angled triangle ABC, having $C=90°$, $CA=CB=1''$.
Calculate length of AB. What is the size of $\angle$A?
Find from the figure sine, cosine, and tangent of 45°.

17. On squared paper draw a large quadrant of a circle AOB, centre O [radius 5 inches, i.e. 10 half-inches]. (Fig. 17.)

By construction divide arc AB into three equal parts at C, D. [Chord AD = radius.]

Then by trial divide each of the arcs AC, CD, DB into three equal parts.

From each of the points of division draw a line perpendicular to OA. Read off the lengths of these lines, thence write down the sines of 10°, 20°, etc., 80°.

18. In the last Example suppose P to be a point on the arc near to A, and let PM be perpendicular to OA. As P moves towards A, how does PM change, and what becomes of it when P reaches A? To what value does the sine of an acute angle continually approach as the angle becomes smaller?

Let Q be a point on the arc near to B, and let QN be perpendicular to OA. As Q moves towards B, how does QN change, and what becomes of it when Q reaches B? To what value does the sine of an acute angle continually approach as the angle gets nearer to 90°? [These results are expressed by saying, **sin 0° = 0, sin 90° = 1.**]

19. From the figure of Ex. **17** write down the cosines of 10°...80°. Check by tables. As in Ex. **18** deduce cos 0° and cos 90°. [**cos 0° = 1, cos 90° = 0.**]

20. If a length equal to the radius of the circle be taken to represent unity, which length in the figure will represent sin 10°, sin 20°, etc.?

21. Using Fig. 17, draw the graph of $y = \sin x°$ from $x = 0$ to $x = 90$. Take $\frac{1}{2}''$ along the x-axis to represent 10° and a distance equal to the radius of the circle to represent 1 along the y-axis. (Fig. 18*.)

22. With the same axes as in Ex. **21** draw the graph of $y = \cos x°$ from $x = 0$ to $x = 90$. [See Exs. VI. 3.] (Fig. 18*.)

23. Does the sine of an acute angle increase more or less rapidly as the angle increases?

24. For what value of x does $\sin x - \cos x = \cdot 5$?

25. On squared paper draw a quadrant of a circle AOB, centre O, radius 1″. As in Ex. **17** divide the arc into nine equal parts. (Fig. 19.)

From A draw AL perpendicular to OA. Join O to each of the points of division on the arc, and produce the joining lines to meet AL. Read off the lengths cut off along AL, thence write down the tangents of 10°, 20°, etc.

* In Exs. **21** and **22**, notice that by bisection of arcs we can easily get the sine and cosine of 25°, 35°, etc., and so get more points on the graphs if desired.

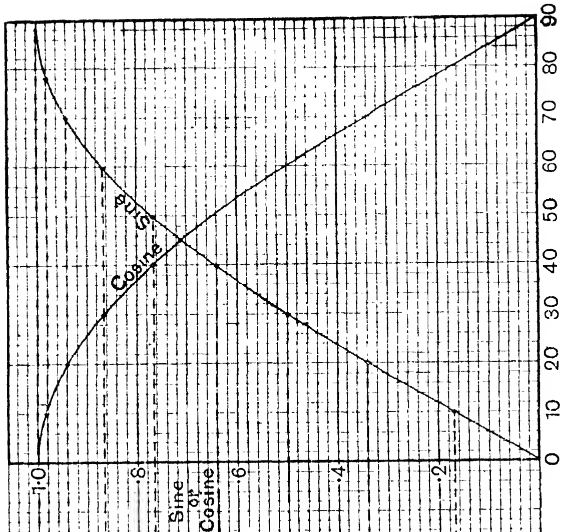

Sine
Cosine
Sine or Cosine
1·0
·8
·6
·4
·2
0
10
20
30
40
50
60
70
80
90
No. of degrees in angle.

Fig. 18.

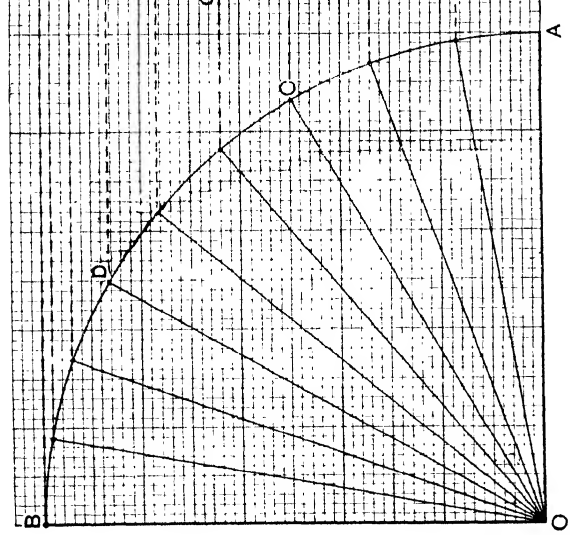
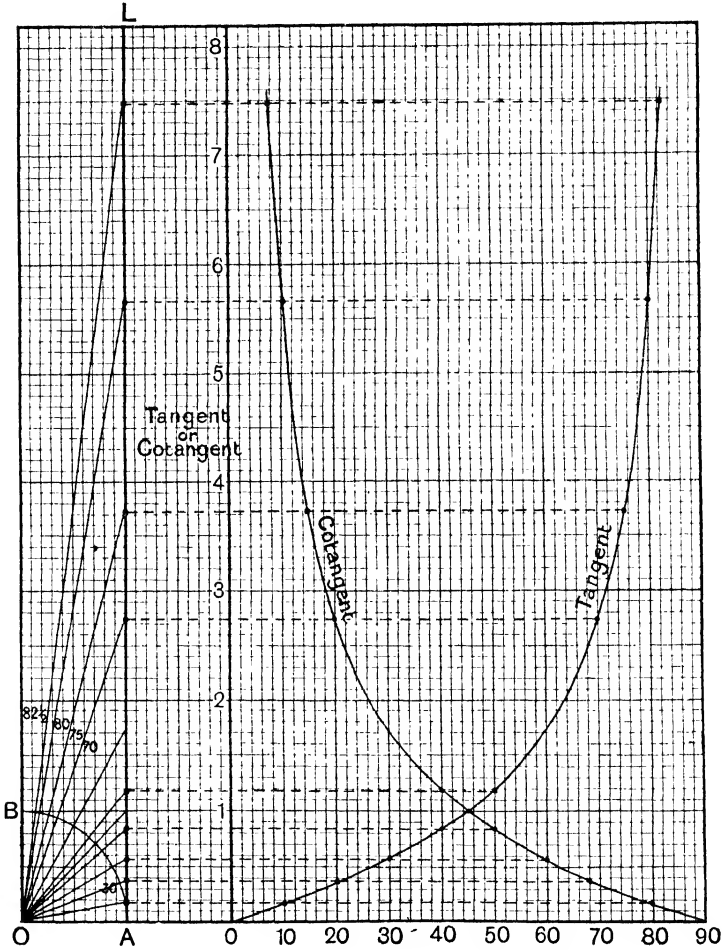

B
D
C
A
O

Fig. 17.



No. of degrees in angle.

Fig. 19.

Fig. 20.

26. In the last Example, suppose P to be a point on the arc near to A, and let OP produced meet AL in p. As P moves towards A, how does Ap change, and what becomes of it when P reaches A? To what value does the tangent of an acute angle continually approach as the angle becomes smaller?

Let Q be a point on the arc near to B, and let OQ produced meet AL in q. As Q moves towards B, how does Aq change? If Q is very near to B, where is q?

What do you conclude about the tangent of an acute angle as the angle gradually approaches 90°?

Look at your table of natural tangents, and notice the values at the end of the table.

[These results are expressed by saying, **tan 0° = 0, tan 90° = ∞**.]

This last symbol is read "infinity," and the statement means that the tangent of an acute angle may be made as large as we please by making the angle near enough to 90°.

27. Use Fig. 19 to draw the graph of $y = \tan x°$ from $x = 0$ to $x = 90$. [Scales as in Ex. **21**.] (Fig. 20.)

28. Does the tangent of an acute angle increase more or less rapidly as the angle increases?

29. Use tables to draw a graph of $y = \sin x°$ between $x = 30$ and $x = 31$.

Use the values given for 30°, 30° 6′, 30° 12′, etc.

Take $\frac{1}{10}$ in. horizontally to represent 1′ and 5 ins. vertically to represent ·01. [Start from ·5.]

From the graph show that, as nearly as you can tell, the difference between sin 30° and sin 30° 3′ is the same as between sin 30° 24′ and sin 30° 27′. What is this difference? Compare with the number given in the difference column. (Fig. 21.)

30. In the same way draw a graph of $y = \tan x°$ between $x = 45$ and $x = 46$.

Take the same horizontal scale and vertically take 1 in. to represent ·01. [Start from 1.]

Test that the differences are appreciably the same in all parts of the graph. (Fig. 22.)

31. Draw a graph of $y = \tan x°$ between $x = 87$ and $x = 88$.

Take the same horizontal scale and vertically take 1 in. to represent 1. [Start at 19.] What difference do you notice between this graph and the last?

Why are no difference columns given in the book at this stage? (Fig. 23.)

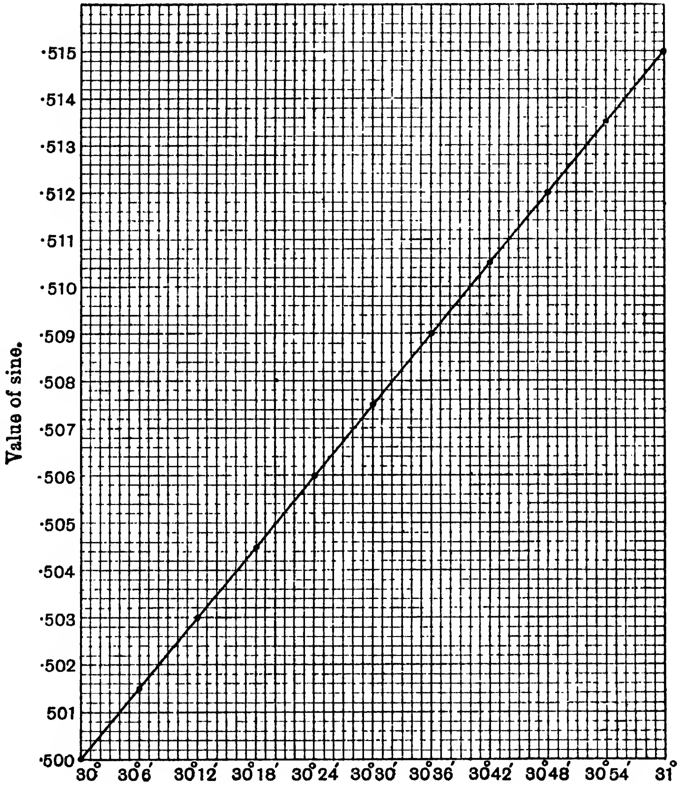


Fig. 21.

Graph of $y=\sin x°$ between $x=30$ and $x=31$.

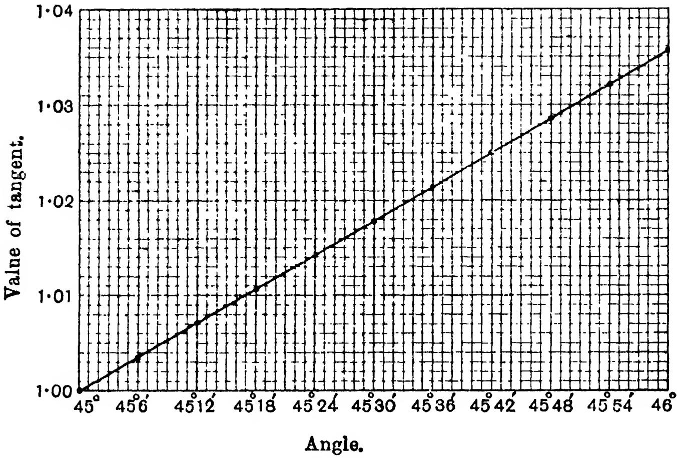


Fig. 22.

Graph of $y = \tan x°$ between $x = 45$ and $x = 46$.

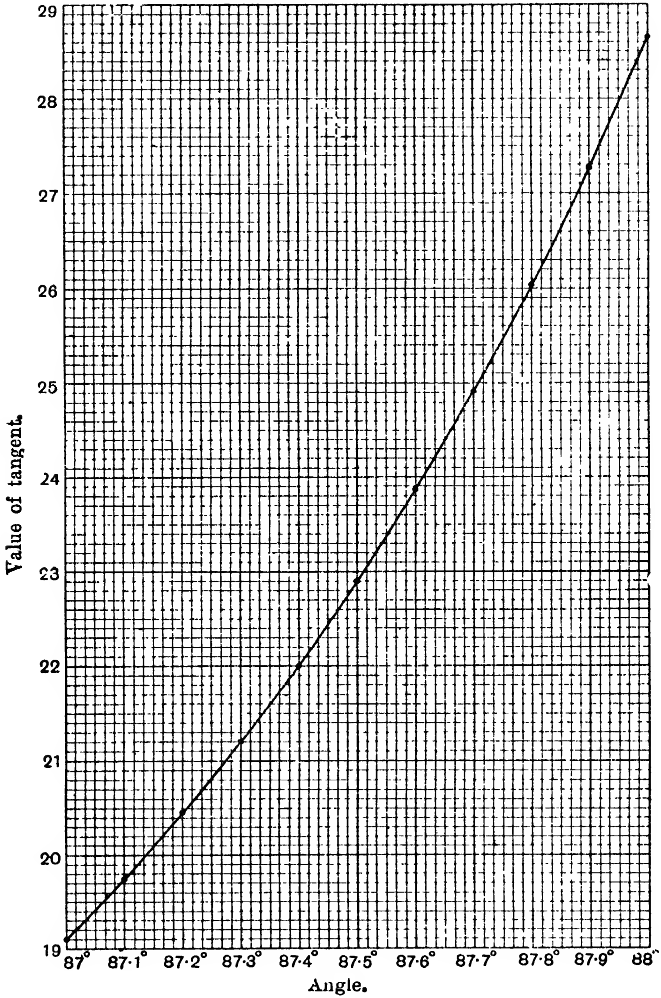


Fig. 23.

Graph of $y = \tan x°$ between $x = 87$ and $x = 88$.

CHAPTER III

SOLUTION OF RIGHT-ANGLED TRIANGLES

29. A triangle is said to be *solved* when all its sides and angles are known.

Any triangle can be solved if we are given one side and two other parts. In particular a right-angled triangle can be solved, if in addition to the right angle we are given two sides or one side and one angle.

Construct with instruments the triangles in which

(i) $C = 90°$, $c = 2{\cdot}5$, $a = 1{\cdot}5$ [hypotenuse and one side].

(ii) $C = 90°$, $a = 1{\cdot}6$, $b = 3$ [two sides containing right angle].

(iii) $C = 90°$, $A = 54°$, $a = 4$ [angle and one of sides containing right angle].

(iv) $C = 90°$, $A = 43°$, $c = 5$ [angle and hypotenuse].

In each case notice that the triangle is determined by the given parts, and measure the remaining parts.

30. In solving triangles by Trigonometry we use the following facts:

If ABC has $C = 90°$ (Fig. 7, § 10),

(i) $c^2 = a^2 + b^2$;

(ii) $A + B = 90°$;

(iii) $\sin A = \dfrac{a}{c}$ or $a = c \sin A$;

(iii′) $\sin B = \frac{b}{c}$ or $b = c \sin B$;

(iv) $\cos A = \frac{b}{c}$ or $b = c \cos A$;

(iv′) $\cos B = \frac{a}{c}$ or $a = c \cos B$;

(v) $\tan A = \frac{a}{b}$ or $a = b \tan A$;

(v′) $\tan B = \frac{b}{a}$ or $b = a \tan B$.

31. To solve the triangles in § 29 by Trigonometry.

(i) (1) First find A.

Use that ratio of A which involves the two given sides.

$$\sin A = \frac{a}{c} = \frac{1{\cdot}5}{2{\cdot}5} = {\cdot}6,$$

$$\therefore\ A = 36^\circ\ 52'.$$

(2) Find B.
$$B = 90^\circ - A$$
$$= 53^\circ\ 08'.$$

(3) Find b. We can do this either directly from the data or by making use of the value of A or B just found.

(α) By Pythagoras' theorem

$$b^2 = c^2 - a^2$$
$$= 2{\cdot}5^2 - 1{\cdot}5^2$$
$$= 4 \times 1,$$
$$\therefore\ b = 2.$$

(β) Use a ratio of A or B which involves b and one of the given sides.

The multiplier which converts c into b is cos A or sin B.

.... a .... tan B.

$\therefore$ we can say either $b = c \sin B = 2{\cdot}5 \times {\cdot}8000 = 2$,

or $b = a \tan B = 1{\cdot}5 \times 1{\cdot}3335 = 2{\cdot}000$.

The advantage of (a) is that we only use quantities given and are therefore independent of any error we may have made in calculating B.

(ii) (1) Find A.
(2) ... B.
(3) ... c by Pythagoras' theorem, or by using a and A.

(iii) (1) Find B.
(2) ... b.
(3) ... c.

(iv) (1) Find B.
(2) ... a.
(3) ... b.

In each of these cases the area is $\frac{1}{2}ab$ and can therefore be found when we have a and b.

32. It is important in all cases to check your results.

The safest check is to take $C = 90°$ and two of the three parts we have found (one at least being a side) and from these calculate the remaining parts. Their values should agree with what we have recorded.

For instance in the first example of § 31 we were given

$$\begin{cases} C = 90°, \quad c = 2{\cdot}5, \quad a = 1{\cdot}5 \text{ and we calculated} \\ A = 36° \, 52', \quad B = 53° \, 08', \quad b = 2. \end{cases}$$

Take $b = 2$, $A' = 36° \, 52'$, $C = 90°$ and find a, c, B.

(1) $A + B = 90°$, $\therefore B = 53° \, 08'$;

(2) $$a = b \tan A$$
$$= 2 \times {\cdot}7500$$
$$= 1{\cdot}5 ;$$

(3) $$c = \frac{b}{\cos A} \text{ or } \frac{b}{\sin B} = \frac{2}{{\cdot}8000} = 2{\cdot}5 ;$$

and these agree with what we previously recorded.

EXERCISES. XII.

1. Complete the solutions to the last three examples in § 31.

Solve the following triangles and find their areas:

2. $C=90°$, $a=3$, $b=4$.
3. $A=90°$, $c=5$, $a=13$.
4. $B=90°$, $A=40°$, $b=5$.
5. $C=90°$, $B=64° 12'$, $b=10$.
6. $A=90°$, $a=61$, $b=11$.
7. $B=90°$, $a=9$, $c=40$.
8. $A=90°$, $B=15° 15'$, $a=12$.
9. $B=90°$, $a=8$, $c=15$.
10. $C=90°$, $A=71° 9'$, $b=8$.
11. $C=90°$, $c=65$, $a=16$.
12. $A=90°$, $B=28°$, $a=7$.
13. $B=90°$, $C=53° 18'$, $a=4$.

Check your answers as indicated above.

Problems.

EXERCISES. XIII.

1. At a point 200 feet from a tower which stands on a horizontal plane the angle of elevation of the top is 58°. Find its height and the angle of elevation of the top at a point 100 feet from the base. Also find at what distance the angle of elevation will be 29°.

2. A kite string is 250 yards long. How high is the kite above the ground when the angle of elevation is 27°?

How much does the kite rise when the angle of elevation changes from 27° to 48°?

3. Two posts 16 and 10 feet high stand 8 feet apart on a level road. Their tops are joined by a tight wire. Find the length of the wire and its inclination to the vertical.

4. Find the altitude of the sun, when a 12 foot pole standing vertically casts a shadow (i) 1 foot, (ii) 4 ft. 6 ins., (iii) 28 ft. long.

5. A stick 12 feet long stands upright. Find the length of the shadow cast by the sun if its altitude is (i) 28° 15′, (ii) 62°, (iii) 10°.

6. Find the angle subtended at a point on the ground by a man 6 feet high at a distance of 100 yards in the same horizontal plane.

7. A trap-door in a roof is 4 feet long from the hinge to the opposite edge. How far is any point of this edge (i) from its original position, (ii) from the roof, when the trap-door makes an angle of 40° with the roof?

8. A picture 2 ft. high hangs by two parallel cords each 4 ft. long. If the picture makes an angle of 15° with the wall, how far is the top edge from the wall, and what angle do the cords make with the wall?

9. In a flight of stairs each step is 1 foot broad and 9 inches high. Find the inclination of the banister rail to the horizontal.

10. A man has a 60° set square. He holds it with the angle 60° to his eye, the short side horizontal, and the long side pointing to the top of a tower. If he is 50 feet from the tower, how high is it, supposing that the height of the man's eye above the ground is 6 feet?

11. A ship and two buoys lie in a straight line. The angles of elevation of the masthead from the buoys are 5° and 11°. If the height of the masthead above the water-line is 150 feet, how far are the buoys apart (i) if they are on the same side, (ii) if they are on opposite sides of the ship?

12. Three ships are anchored in a line with the foot of a cliff 250 feet high. An observer on the top of the cliff finds the angles of depression to be 30°, 24°, 12° 30′. Find the distance of the middle ship from each of the others.

13. Two ships sail from a port together. A sails due S., and after B has sailed S. 20° E. 70 miles, A bears due W. Find their distance apart.

14. A port bears from a ship N. 20° E. 10 miles; what will be its bearing and distance after the ship has sailed S. 70° E. 20 miles?

15. A point of land distant 10 miles from a ship bore N.; what were its bearing and distance after she had sailed due E. 12 miles?

16. A target is to be laid 1400 yards from a ship whose masthead is 180 feet above the water-line. What will be the angle of elevation of the masthead from the position of the target?

Definition. A **knot** is a speed of a nautical mile (about 6080 feet) per hour. In questions involving knots it is understood that "mile" means "nautical mile."

17. A steamer which can do 12 knots observes a ship bearing N. 36° 52′ E. travelling due W. If the ship be 5 miles off and sailing at 9 knots, show that the steamer will come up with her in about 20 mins. if she steams due N. at full speed.

18. A man whose eye is 5 ft. 6 ins. above the ground stands in front of a vertical mirror and finds that the angle of depression of his feet as seen in the mirror is $15\frac{1}{2}$°. How far is he from the mirror?

[The image of the man is as far behind the mirror as he is in front.]

19. A window in a house faces S. On the other side of the street is a tall building above which just half the sun appears at noon on Dec. 22nd. Find the height of the building above the window, given that width of street is 30 feet, and altitude of sun at noon on Dec. 22nd is 15° [i.e. altitude of sun's centre].

20. From a lower window in the house, just half the sun is seen above the same building at noon on June 22nd. Find the vertical distance between the windows if the altitude of the sun at noon on June 22nd is 62°.

21. In a simple engine the connecting rod is $3\frac{1}{2}$ feet long and the crank 1 foot long. Find the angle between the crank and the line of dead centres at $\frac{1}{2}$ stroke.

22. Also find the distance of the cross-head from the centre of the crank circle, when the crank makes an angle of 90° with the line of dead centres. What is the angle between the crank and the connecting rod?

23. A small weight hangs from a nail in a vertical wall by a string 2 ft. 6 ins. long. The weight is drawn aside until it is 1 foot from the wall. What is the inclination of the string to the vertical?

24. A tangent is drawn to a circle of radius $1''$, from a point O, $3''$ from the centre C. Find the length of the tangent and the angle it makes with OC.

25. OA, OB are two radii of a circle containing an angle of 50°. AC is perpendicular to OB. Find the length of BC, given that the radius of the circle is 5 inches.

26. AB is a diameter of a circle. PQ is a chord perpendicular to AB, cutting AB in N. If the angle PAN $=35^\circ$, find lengths of PA, PB, PQ, given that AB $=10$ cms.

27. AB is a diameter of a circle. PNQ is a chord perpendicular to AB. If AB $=10$ cms., and PQ $=6$ cms., find the angle PAN.

28. ABCDE is a regular pentagon of side $1''$. Find the length of AC.

29. ABC is an isosceles triangle in which B $=$ C $=70^\circ$. P is any point in BC, PX and PY are perpendicular to AB and AC respectively. Show that PX + PY is the same wherever P is in BC. [Call the lengths BP and PC, p'' and q'' respectively.]

30. A crane has a vertical crane post AB 8 feet long, and a horizontal tie BC 6 feet long, AC being the jib. Find the length of the jib and the angle it makes with the crane post.

31. A crane has a vertical crane post 8 feet long, a tie 8 feet long, and a jib 12 feet long. Find the angles made with the post by the tie and jib.

32. Find the resultant of two forces, 2 lbs. wt. and 3 lbs. wt., acting at an angle of 90°, and find the angle it makes with each force.

33. Find the resultant of two equal forces each of 10 lbs. wt. acting at an angle of (i) 40°, (ii) 160°.

34. Forces of 4, 4, 5 lbs. wt. acting at a point are in equilibrium. Find the angles between their directions.

35. A is 20 miles N.W. of B. C is 30 miles N.E. of B. Find the bearing of C from A. [Project on N.-S. and E.-W. lines.]

36. B is 15 miles N. 65° E. of A. C is 40 miles N. 35° W. of B. Find the bearing of A from C.

37. In $\triangle ABC$, $AB=10$ cms., $BC=12$ cms., $\angle ABC=60°$.

AD is perpendicular to BC.

Find AD and BD. Hence DC.

Then from $\triangle ADC$ find AC and the angle C.

38. In $\triangle ABC$, $AB=10$ cms., $\angle ABC=60°$, $\angle ACB=53° 08'$.

AD is perpendicular to BC.

Find AD and then from $\triangle ADC$ find AC.

Similarly by drawing BE perpendicular to AC, find BC.

39. A rock is 2000 yards from a cliff 350 feet high. What is the angle of elevation of the top of the cliff from the rock? If a destroyer runs so that the angle of elevation of the cliff is 3°, by how much will it clear the rock?

40. A rock is 1000 yards from a cliff 300 ft. high. What must be the angle of elevation of the cliff from a destroyer, so that it may clear the rock by 200 yards?

41. A and B are two prominent landmarks 4000 yards apart. B is S. 42° E. from A. From a ship, A bears due N. and B due E. How far is the ship from each landmark?

42. A and B are two landmarks 3000 yards apart, B being S. 20° E. from A. From a ship, A bears N. 30° E. and B bears S. 60° E. Find the distances of the ship from A and B.

43. Wishing to find the breadth of a river, I measure a length AB of 100 yards close to one bank. T is a tree on the opposite bank. I measure the angles BAT and ABT. They are 60° and 30° respectively. Find the breadth of the river. [First find AT.]

44. ABCD is a kite in which AB = AD = 5″, CB = CD = 8″, BD = 6″. Find AC and the angles of the kite.

45. From the top of a building 50 feet high, the angles of depression of the top and bottom of a flagstaff stuck in the ground are 10° and 15°. Find the height of the flagstaff and its distance from the building.

46. Wishing to know the height of a window, I measure 50 feet away from the foot of the building and find the angles of elevation of the top and bottom of the window to be 43° and 38½°. Find the height of the window.

33. The following example shows how to deal with a case of frequent occurrence:

A man wishes to find the height of an inaccessible tower. He finds the angle of elevation to be 35°. He comes 100 ft.

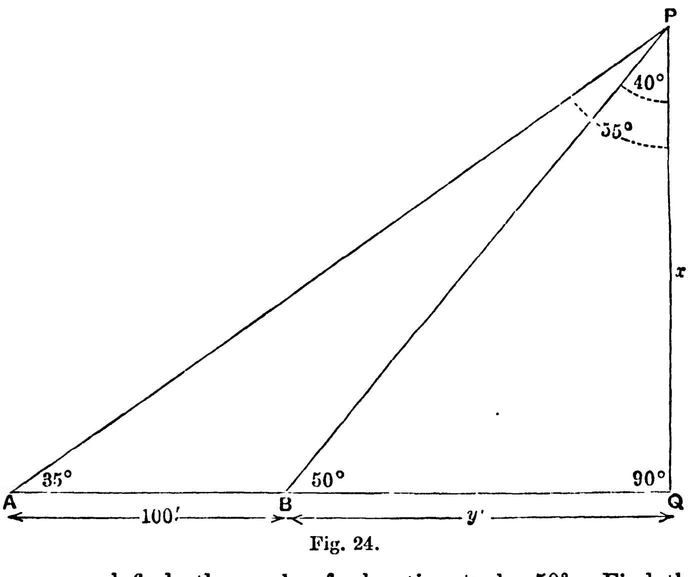


Fig. 24.

nearer, and finds the angle of elevation to be 50°. Find the height of the tower and its distance from the second position.

PQ is the tower.

A and B the two positions of the observer.

Let PQ be x' and BQ y'. Then AQ is $(100+y)'$.

From $\triangle$PQB

$$y = x \tan 40^\circ = x \times \cdot 8391 \ldots\ldots\ldots\ldots (1).$$

From $\triangle$PQA

$$100 + y = x \tan 55^\circ = x \times 1\cdot 4281 \ldots\ldots\ldots\ldots (2).$$

$\therefore$ by subtraction

$$100 = x \times \cdot 5890,$$

$$\therefore\ x = \frac{100}{\cdot 5890}$$

$$= 169\cdot 8.$$

$$\begin{array}{r} 169\cdot 8 \\ 589\,)\,\overline{100000} \\ \overline{4110} \\ \overline{5760} \\ \overline{4590} \end{array}$$

[Or we might use a table of reciprocals]

$\therefore$ to nearest foot

height of tower = 170 feet.

To get y we can substitute this value of x in (1) *or* we can find y independently as follows:

From $\triangle$PQB

$$x = y \tan 50^\circ = y \times 1\cdot 1918.$$

From $\triangle$PQA

$$x = (y + 100) \tan 35^\circ = (y + 100) \times \cdot 7002,$$

$$\therefore\ y \times 1\cdot 1918 = (y + 100) \times \cdot 7002$$

$$= \cdot 7002y + 70\cdot 02,$$

$$\therefore\ \cdot 4916y = 70\cdot 02,$$

$$\therefore\ y = \frac{70\cdot 02}{\cdot 4916}$$

$$= 142\cdot 4.$$

$$\begin{array}{r} 142\cdot 4 \\ 4916\,)\,\overline{700200} \\ \overline{20860} \\ \overline{11960} \\ \overline{21280} \end{array}$$

$\therefore$ to nearest foot

BQ = 142 feet.

EXERCISES. XIII (*continued*).

47. A man observes the angle of elevation of a tower to be 30°. He comes 50 yards nearer and finds the angle of elevation to be 70°. Find the height of the tower.

48. From the top of a hill two consecutive milestones are seen in a straight road which runs from the foot of the hill. Their angles of depression are 5° and 15° respectively. What is the height of the hill?

49. A and B are two points on a straight road running N. 50° E. B is 300 yards from A. The bearing of a certain house from A is N. 17° E., and from B N. 10° W. Find the distance of the house from the road.

50. A flagstaff 20 feet long is placed on the top of a tower. At a point in the horizontal plane through the foot of the tower, the tower subtends an angle of 45°, and the flagstaff an angle of 9°. Find the height of the tower.

51. Find the area of a triangle whose base is 10 inches, base angles 28° and 116°.

52. In Fig. 24 if $\angle QAP=35°$, $\angle QBP=70°$, $AB=100$ feet, what is BP? Hence find BQ and QP.

53. From the masthead of a ship the angles of depression of two buoys lying in the same direction are 10° and 20°. If the buoys are 100 yards apart, what is the height of the masthead?

54. A man walking due E. along a straight road finds that a church spire bears N. 60° E. A mile further it bears N. 30° E. How far is the church N. of the road?

55. Find the area of a triangle whose base is 10 inches, base angles 35° and 110°.

56. A, B, C are three stations lying in a straight line in the same horizontal plane as Q the foot of a tower PQ. The angles of elevation of the tower from A, B, C are 50°, 35°, 25° respectively. If AB is 60 feet, find BC. ABCQ is a straight line.

57. The angles of elevation of the top of a flagstaff from two points on opposite sides of it are 30° and 60°. If the points are 200 feet apart, find the height of a flagstaff and its distance from each point.

58. From a ship steaming due E. at 15 knots, the bearing of a lighthouse at 1 o'clock is N. 53° E. 40 minutes later the bearing is N. 40° W. Find the least distance of the ship from the lighthouse, and at what time it will be at this least distance.

59. A ravine is spanned by a horizontal bridge 100 yards long. If the sides of the ravine are inclined at angles 40° and 30° to the horizontal, what is the depth of the lowest point of the ravine below the bridge?

60. Find the area of a triangle, given

(i) Base 10 ins.; Base angles 55° and 35°;

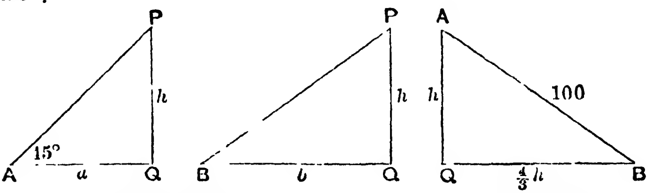
(ii) Base 10 ins.; Base angles 42° and 58°.

34. Example of a case in which all the points involved are not in the same plane.

From A due North of a tower PQ the angle of elevation of P, the top of the tower, is 45°. From B due East of the tower the angle of elevation is 36° 52′. If A, B, Q are in a horizontal plane and if AB = 100 ft., find the height of the tower.

[Take $\tan 53° 08' = \frac{4}{3}$.]

Here we have three right-angled triangles as shown in the figure*.



Vertical triangles. Ground plan.

Fig. 25.

From the first triangle $a = h \tan 45° = h$.

...........second ...... $b = h \tan 53° 08' = \frac{4}{3}h$.

...........third......... $a^2 + b^2 = 100^2$.

$$\therefore\ h^2\left(1 + \frac{16}{9}\right) = 100^2,$$

$$\therefore\ h^2 = 100^2 \times \frac{9}{25},$$

$$\therefore\ h = 100 \times \frac{3}{5} = 60.$$

$\therefore$ the height of the tower is 60 ft.

* A paper model may be made as follows:—First draw Δs PQA, PQB taking any convenient length to represent the height of the tower. From any point Q draw lines due N. and due E. equal to QA and QB respectively (third figure). Then on the same scale on which AB represents 100 feet, AQ represents the height of the tower. Draw to the left QP_1 equal to QP and perpendicular to QA. Draw below QB, QP_2 equal to QP and perpendicular to QB. [These Δs AQP_1 and BQP_2 are merely copies of AQP and BQP.] Cut through the lines AP_1, P_1Q, QP_2, P_2B and using AQ and BQ as hinges bring QP_1 into coincidence with QP_2.

EXERCISES. XIII (*continued*).

61. From a station A, a hill bears N.E. and the angle of elevation of the top of the hill is 45°. From B, 500 feet E. of A, the hill bears N.W. Find the height of the hill.

62. From A, due S. of a tower, the angle of elevation of the top of the tower is 45°. From B, 100 feet due E. of A, the angle of elevation is 36° 52′. Find the height of the tower. $\left[\text{Take } \tan 53^\circ 8' = \frac{4}{3}.\right]$

63. From A, due W. of a tower, the angle of elevation of the top is 54°. What will be the angle of elevation from B which is due S. of A and S. 30° W. from the tower?

64. A ship is 1200 yards due W. of a cliff 500 ft. high. What will be the angle of elevation of the cliff when the ship has sailed 900 yards S.?

65. A, B are respectively North and East of a tower whose height is 100 feet and are in the same horizontal plane as the base of the tower. If the angles of elevation of the tower from A and B are 18° and 24°, find the distance and bearing of B from A.

66. A straight road is 30 feet wide. Two vertical posts of equal height are placed 40 feet apart on one side of the road. From a point on the other side of the road directly opposite one post, the angle of elevation of this post is 30°. What is the angle of elevation of the other post from the same point?

67. An aeroplane is flying at a constant height due East at 40 miles per hour. An observer takes its elevation when it is due North of him. It is 45°. A minute later the elevation is 18° 26′. What is the height of the aeroplane? $\left[\tan 18^\circ 26' = \frac{1}{3}.\right]$

68. A flagstaff 30 feet long leans over towards the East at an angle of 20° to the vertical. Find the length of the shadow and the direction in which it points if the sun is due South at an altitude of 45°.

69. The same as Ex. **68** with sun's altitude 36° 52′ instead of 45°.

70. An aeroplane is observed simultaneously from two ships A and B. From A it bears due N. and its elevation is 26° 34′. From B it bears due E. and its elevation is 18° 26′. If the distance between the two observers is 2000 yards find the height of the aeroplane. $\left[\tan 26^\circ 34' = \frac{1}{2}.\right]$

CHAPTER IV

OTHER TRIGONOMETRICAL RATIOS—COTANGENT, SECANT, COSECANT

35. *Definition.* The reciprocals of the sine, cosine, and tangent of an angle are called respectively the cosecant, secant, and cotangent.

The abbreviations for these are cosec, sec, and cot.

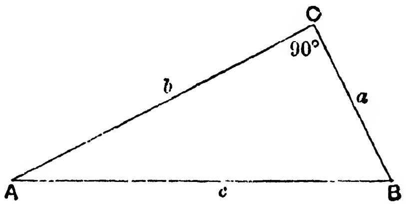


Fig. 26.

If ABC be a triangle in which C is a right angle

$$\sin A = \frac{a}{c}\left(\frac{\text{opp. side}}{\text{hyp.}}\right),$$

$$\cos A = \frac{b}{c}\left(\frac{\text{adj. side}}{\text{hyp.}}\right),$$

$$\tan A = \frac{a}{b}\left(\frac{\text{opp. side}}{\text{adj. side}}\right),$$

$$\cot A = \frac{1}{\tan A} = \frac{b}{a} = \left(\frac{\text{adj. side}}{\text{opp. side}}\right),$$

$$\sec A = \frac{1}{\cos A} = \frac{c}{b} = \left(\frac{\text{hyp.}}{\text{adj. side}}\right),$$

$$\operatorname{cosec} A = \frac{1}{\sin A} = \frac{c}{a} = \left(\frac{\text{hyp.}}{\text{opp. side}}\right).$$

36. As in §§ 10 and 24 these new ratios may be defined as multipliers. Thus

sec A is the multiplier which converts the side adjacent to A into the hypotenuse,

cosec A is the multiplier which converts the side opposite to A into the hypotenuse,

cot A is the multiplier which converts the side opposite to A into the side adjacent to A.

37. Though, as will be seen, it is often convenient to use these three ratios, we can get on without them. All the exercises in this chapter except those in which cot, sec, cosec are explicitly mentioned can be solved by the use of sine, cosine and tangent.

38. To illustrate the use of these new ratios, consider the example:

In a triangle, $C = 90°$, $A = 40°$, $a = 10$, find c.

The method we have previously used is

$$a = c \sin A,$$

$$\therefore \quad c = \frac{a}{\sin A} = \frac{10}{\sin 40°} = \frac{10}{\cdot 6428} = 15{\cdot}55.$$

Here we used $\sin A$, the multiplier which converts the unknown side c into the known side a, and had to rewrite the relation so as to get c in terms of a and A. If we use the multiplier which converts a (the known) into c (the unknown) we get

$$\begin{aligned} c &= a \operatorname{cosec} A \\ &= 10 \times 1{\cdot}5557 \\ &= 15{\cdot}557. \end{aligned}$$

EXERCISES. XIV.

1. Look out sin 35° and cosec 35°, and verify $\frac{1}{\sin 35°} = \text{cosec}\, 35°$.

2. Similarly verify

$$\cos 60° = \frac{1}{\sec 60°}, \quad \cot 42° = \frac{1}{\tan 42°}, \quad \sec 75° = \frac{1}{\cos 75°}.$$

3. Find by drawing:

(i) the angles whose cotangents are ·37, 2·81;

(ii) the angles whose secants are 3·14, 1·29;

(iii) the angles whose cosecants are 4, 3·14.

Check your results from the tables.

4. Solve the triangles in which

(i) C = 90°, A = 38°, a = 5;

(ii) B = 90°, C = 64°, a = 8;

(iii) A = 90°, B = 80°, b = 10;

(iv) A = 90°, B = 51°, c = 12;

(v) B = 90°, A = 47°, a = 2;

(vi) C = 90°, B = 72°, a = 20.

5. Solve the triangles in which

(i) a = 100, B = C = 80°;

(ii) b = 10, A = C = 54°;

(iii) c = 16, A = B = 15°;

(iv) a = 8 = b, A = 41°;

(v) c = 20, a = b, C = 48°;

(vi) b = 10, a = c, A = 70°;

(vii) a = 10, b = 8, B = C;

(viii) b = 6 = c, A = 53°.

6. A is 20 miles W. of B. A ship sails from A until she is due N. of B. If her course is N. 38° E., how far has she sailed?

7. A point of land bore from a ship N. 35° W. After she had sailed S. 55° W. 20 miles it bore N.E. What was her distance from it at each observation?

8. An object on shore bears from a ship N. 34° W. After the ship has sailed N. 56° E. 8 miles, the object bears N. 69° W. Find the distance of the object from the ship at the second observation.

9. A is 1000 yards N. 40° W. from B. From C, A bears N. 50° E. and B bears N. 70° E. Find the distances of C from A and B.

10. From the top of a cliff 125 feet high a man observes the angle of depression of a boat to be 35° Find the distance of the boat (i) from the foot of the cliff, (ii) from the man.

11. In a triangle ABC, AC = 10 cms., C = 70°, B = 30°. AD is perpendicular to BC. Find AD, and then AB.

Similarly find BC by drawing CF perpendicular to AB.

12. In the triangle ABC, the altitude AD = 5 cms., B = 43°, C = 70°. Find the lengths of the sides of the triangle.

13. In the triangle ABC, the altitude AD = 8 cms., B = 32°, C = 123°. Find the lengths of the sides.

14. In a triangle ABC, B = 30° 48′, C = 52° 7′. AD perpendicular to BC is 1″. By how much is AB + AC greater than BC?

15. Same as Ex. **14** given B = 29°, C = 123° 8′, AD = 1″.

16. ABCDE is a regular pentagon. AF is perpendicular to CD. If AF = 10 cms., find AC and AB.

17. A chord 10″ long subtends an angle of 84° at the centre of a circle. What is the length of the radius?

18. BC the base of a triangle is 4 ins. long. The vertical angle A is 70°. Find the radius of the circle passing through A, B, and C.

[Let O be the centre. What is the angle BOC?]

19. Find the radius of the circle ABC (i) when BC = 5″, A = 54°, (ii) BC = 10 cms., A = 110°.

20. Two tangents from a point O to a circle of radius 2″ contain an angle of 42°. Find their lengths, and the distance of O from the centre.

21. O is a point distant 3·8″ from the centre of a circle of radius 2″. Find the angle between the tangents from O to the circle.

22. Find the side of a rhombus whose longer diagonal is 12 cms. and acute angle 48°.

23. A ladder leans against a wall at an angle of 32° to the vertical. Its foot is 20 feet from the wall. Find its length.

24. The string of a kite is inclined at 46° to the vertical. How much string has been let out if the kite is 200 feet above the ground?

25. A man walks up a hill inclined at an angle of 10° to the horizon. How far must he walk to rise 100 feet?

26. In Ex. **25** what is the projection on the horizontal plane of the distance walked by the man?

27. D is the mid-point of the side AB of an acute-angled triangle ABC, in which CA>CB. CE is perpendicular to AB. If AB=14″, EB=3″, CE=5″, show that $\cot A - \cot B = 2 \cot CDB$.

28. With the triangle of Ex. **27**, if K be a point in AB such that $AK = \frac{1}{3} AB$, show that $2 \cot A - \cot B = 3 \cot CKB$.

29. In a trapezium ABCD, AB is parallel to CD, A=55°, B=42°, CD=12 cms., and the distance between AB and CD=10 cms. Find the lengths of AB, AD, BC and the area of the trapezium.

30. In a triangle ABC, C=90°, $a=3$, $b=4$. What is the other side?

Write down the values of sec A and tan A; and find the value of $\sec^2 A - \tan^2 A$.

31. Same as Ex. **30**, taking

(i) $a=5$, $b=12$; (ii) $a=7$, $b=24$; (iii) $a=33$, $c=65$.

32. Find from tables $\sec^2 A - \tan^2 A$ when A=(i) 39°, (ii) 58°, (iii) 71°.

33. Prove $\mathbf{\sec^2 A - \tan^2 A = 1}$, where A is any acute angle. [Use a right-angled triangle in which the side adjacent to A contains one unit of length.]

34. In Fig. 26, § 35, what are

tan A, cot A, sec A, cosec A;
tan B, cot B, sec B, cosec B?

Hence **tan of an angle = cot of its complement, etc.**

Write down the other relations.

35. Verify from tables the statements of Ex. **34** when the first angle is 25°, 33°, 56°, 79°.

36. If $\sin A = \frac{3}{5}$, find without tables cos A, tan A, cot A, sec A, cosec A. Verify by tables. [See Exs. XI. 11, p. 28.]

37. If $\tan A = \frac{5}{12}$, find all the other ratios of A as in Ex. **36**.

38. What happens to the value of cot A, as A gradually approaches (i) zero, (ii) a right angle? [See Exs. XI. 26.]

The same for sec A and cosec A.

39. Draw the graph of $y = \cot x°$ from $x=0$ to $x=90$ on the same paper as the graph of $y = \tan x°$, using the same scales. [See Exs. XI. 27 and Fig. 20.]

40. Using Fig. 19, if a length equal to the radius of the circle be taken to represent unity, what length in the figure will represent sec 10°, sec 20°, etc.?

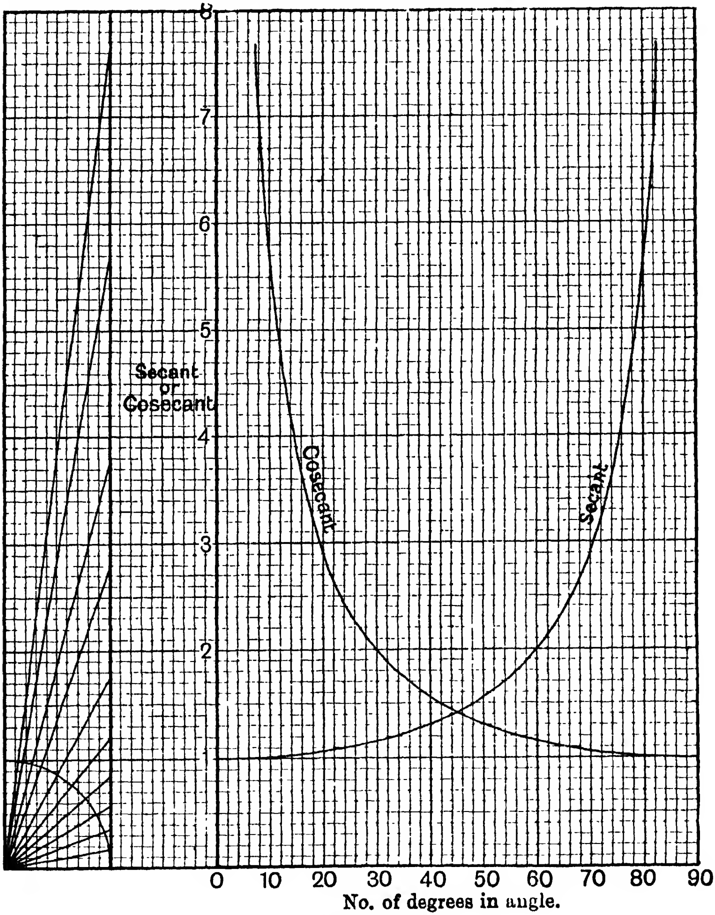


Fig. 27.

41. On the same sheet, draw the graphs of $y = \sec x°$ and $y = \operatorname{cosec} x°$ from $x = 0$ to $x = 90$, using the same scales as in Ex. **39**. (Fig. 27.)

EXERCISES. XV.

Regular Polygons.

1. AB is the side of a regular pentagon *inscribed in* a circle of radius 5 cms., centre O. (Fig. 28.)

What is $\angle$AOB?

Draw OP perpendicular to AB.

Find length of AP. Hence of AB.

Hence perimeter of pentagon.

2. Make a table like the following and fill in the blanks:

No. of sides of inscribed polygon	Angle subtended by side at centre	Length of side	Perimeter of Polygon	$\frac{\text{Perimeter}}{\text{Diameter}}$
3				
4				
5	72°	5·878	29·390	2·939
6				
8				
10				
20				
40				
100				

The radius of the circle is 5 cms.

3. AB is the side of a regular pentagon circumscribed to a circle of radius 5 cms., centre O. OP is the radius to the point of contact. (Fig. 29.)

What is $\angle$AOB?

Find length of AP. Hence of AB.

Hence perimeter of pentagon.

4. Make a table like that in Ex. **2** for circumscribed polygons.

5. What can you deduce from Exs. **2** and **4**, about the ratio of the circumference of a circle to its diameter?

6. Obviously the perimeters of an inscribed 40-gon and an inscribed 100-gon are not equal. Why do your last two results in Ex. **2** agree?

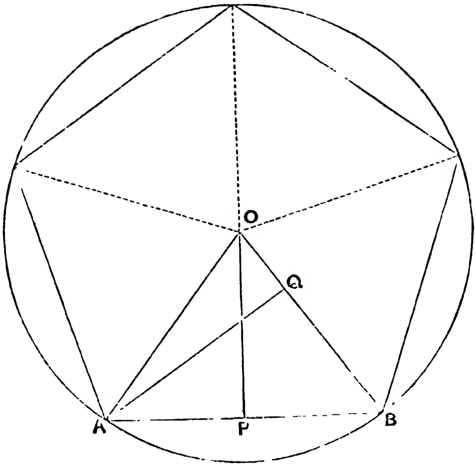


Fig. 28.

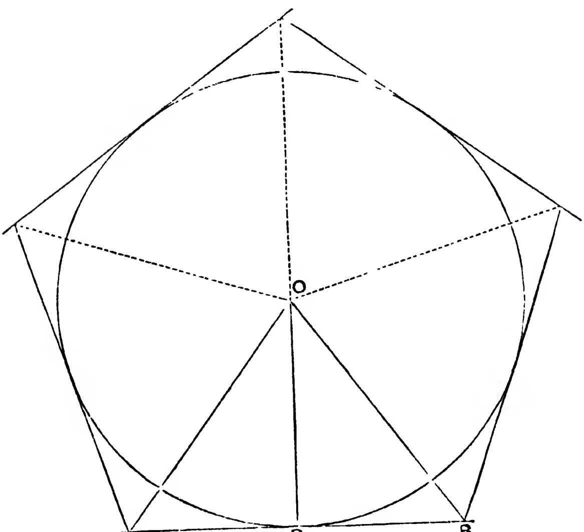


Fig. 29.

7. AB is the side of a regular pentagon inscribed in a circle radius 10 cms., centre O.

AQ is perpendicular to OB. Find length of AQ (p). Hence find area of $\triangle$AOB. Hence of pentagon (*v.* Fig. 28).

8. Make a table as in Ex. **2** with columns headed:

No. of sides of inscribed polygon	p	Area $\triangle$AOB	Area of polygon	$\dfrac{\text{Area of polygon}}{(\text{Radius})^2}$

9. AB is the side of a regular pentagon circumscribed to a circle radius 10 cms, centre O. OP is the radius to the point of contact. Find AB. Hence area $\triangle$AOB.

Hence area of pentagon (*v.* Fig. 29).

10 Make a table as in Ex. **8** for circumscribed polygons.

11. What can you deduce from Exs. **8** and **10** about the ratio of the area of a circle to the square on its radius?

12. Find the lengths of the radii of circles circumscribed to regular polygons of 3, 4, 5, 6, 8, 10, 20, 40, 100 sides, a side of the polygon being 10 cms.

13. Find the lengths of the radii of circles inscribed in the regular polygons of Ex. **12**.

14. Find the areas of the polygons of Ex. **12**.

15. Regular polygons of 3, 4, 5, 6, 8, 10, 20, 40 sides have all the same perimeter, viz. 240 cms. Find their areas.

16. What would you deduce from Ex. **15** as to the greatest area that can be enclosed by a given length of fencing?

17. Regular polygons of 3, 4, 5, 6, 8, 10, 20, 40 sides have all the same area—240 sq. cms. Find their perimeters.

18. Draw graphs showing how the perimeter of a regular polygon (i) inscribed, (ii) circumscribed to a given circle changes as the number of sides is increased.

19. Draw graphs showing how the area of a regular polygon (i) inscribed, (ii) circumscribed to a given circle changes as the number of sides is increased.

CHAPTER V

SOLUTION OF TRIANGLES WHICH ARE NOT RIGHT-ANGLED

39. Every triangle has six parts, three sides and three angles.

The angles are usually denoted by A, B, C and the number of units of length in the sides opposite to them by a, b, c.

Solving a triangle means finding the remaining parts when certain parts are given.

We know from Geometry that if we are given

(i) two sides and the included angle
or (ii) two angles and a side
or (iii) three sides

we can only construct *one* triangle with the given parts.

If we are given

(iv) two sides and the angle opposite to one of them we can construct sometimes one and sometimes two triangles.

We shall find that in each of these cases the triangle can be solved by treating it as the sum or difference of two right-angled triangles which can be solved in turn.

40. In this chapter the following values may be used for the sines and cosines of certain angles to simplify the arithmetic. That these values are very approximately correct should be verified from the tables.

Angle	Sine	Cosine	
11° 32′	$\frac{1}{5}$		78° 28′
14° 29′	$\frac{1}{4}$		75° 31′
19° 28′	$\frac{1}{3}$		70° 32′
22° 37′	$\frac{5}{13}$	$\frac{12}{13}$	67° 23′
30°	$\frac{1}{2}$		60°
36° 52′	$\frac{3}{5}$	$\frac{4}{5}$	53° 8′
41° 24′		$\frac{3}{4}$	48° 36′
41° 48′	$\frac{2}{3}$		48° 12′
	Cosine	Sine	Angle

Any other sines or cosines must be taken from the tables in the usual way.

Case 1. Given two sides and the included angle.

EXERCISES. XVI.

1. Draw as accurately as you can a triangle in which $a=12$, $b=10$, $C=36°\ 52'$ [unit 1 cm.]. Notice that these parts fix the triangle. Measure the remaining parts.

2. In the triangle of Ex. **1** draw AD perpendicular to BC. (Fig. 30.)
Call AD, BD, CD, p, y, z cms. respectively.
In $\triangle$ADC what parts do you know?
Find p and z and hence y.
In $\triangle$ADB what parts do you know?
Find c and B. You can now find A (i.e. BAC).
Make a list of your results $c=$, A= , B= , and compare with what you got in Ex. **1**.

3. What is the area of the triangle in Ex. **2**?

4. Could you solve the triangle by drawing a perpendicular from B to AC or from C to AB? If so, do it; if not, say why.

5. Go through the same steps as in Exs. **1**, **2**, **3**, taking $a=28$, $b=26$ [unit $\frac{1}{2}$ cm.], $C=67°\,23'$.

6. If when you calculate CD it turns out to be greater than CB, what does this show?

E.g. take $a=7$, $b=10$, $C=36°\,52'$.

7. Draw a triangle in which $a=12$, $b=10$, $C=126°\,52'$ [unit 1 cm.], and measure the remaining parts.

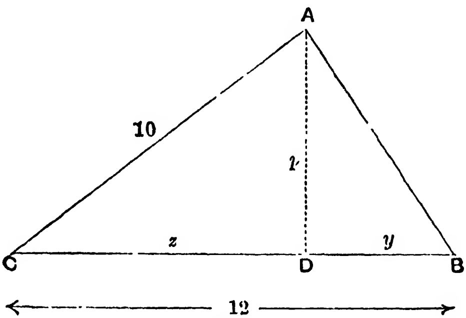


Fig. 30.

8. In the triangle of Ex. **7** draw AD perpendicular to BC produced.

Call AD, BD, CD, p, y, z cms. respectively.

What is $\angle$DCA? In $\triangle$ADC, find p and z and hence y.

In $\triangle$ADB find c and B; and then find A.

Compare your results with what you got in Ex. **7**.

9. Solve the triangle by drawing another perpendicular instead of AD. Also find its area.

10. Go through the same steps as in Exs. **7**, **8**, taking

$$a=28,\quad b=26,\quad C=112°\,37'.$$

41. We will now work out the example in Exs. XVI. 2 to show the arrangement of the work (see Fig. 30).

From $\triangle$ ADC,

$$p = 10 \sin 36° 52'$$
$$= 10 \times \frac{3}{5}$$
$$= 6,$$
$$x = 10 \cos 36° 52'$$
$$= 10 \times \frac{4}{5}$$
$$= 8,$$
$$\therefore \quad y = 4.$$

From $\triangle$ ADB,

$$\tan B = \frac{p}{y} = \frac{6}{4} = 1{\cdot}5,$$
$$\therefore \quad B = 56° 19'.$$

$$c = y \sec B \left(\text{or } \frac{y}{\cos B}\right) \quad or \quad c^2 = y^2 + p^2$$
$$= 4 \times 1{\cdot}8028 \qquad\qquad = 16 + 36$$
$$= 52,$$
$$\therefore \quad c = 7{\cdot}211.$$

$$\left.\begin{aligned} c &= 7{\cdot}211 \\ A &= 86° 49' \\ B &= 56° 19' \end{aligned}\right\} \text{Ans.}$$

This should be checked by solving the triangle in which

$$c = 7{\cdot}211, \quad A = 86° 49', \quad B = 56° 19'.$$

[See next case.]

42. The area of the triangle is now easily found, for it is

$$\frac{1}{2} ap \qquad \text{units of area}$$
$$= \frac{1}{2} \times 12 \times 6 \quad \ldots\ldots\ldots\ldots\ldots$$
$$= 36 \qquad \ldots\ldots\ldots\ldots\ldots$$

EXERCISES. XVII.

Solve the following triangles and find the area in each case [table, § 40]:

1. $b=15$,	$c=13$,	$A=53°8'$.		**2.** $b=8$,	$c=5$,	$A=60°$.	
3. $a=26$,	$b=30$,	$C=22°37'$.		**4.** $a=15$,	$b=16$,	$C=36°52'$.	
5. $b=20$,	$c=12$,	$A=53°8'$.		**6.** $a=26$,	$c=20$,	$B=67°23'$.	
7. $b=6$,	$c=4$,	$A=70°32'$.		**8.** $a=9$,	$b=6$,	$C=48°12'$.	
9. $b=3$,	$c=5$,	$A=120°$.		**10.** $a=25$,	$b=16$,	$C=143°8'$.	
11. $c=4$,	$a=13$,	$B=112°37'$.		**12.** $b=6$,	$c=4$,	$A=109°28'$.	
13. $a=3$,	$b=4$,	$C=52°$.		**14.** $a=2$,	$b=3$,	$C=130°$.	

Case 2. Given two angles and one side.

EXERCISES. XVIII.

1. Draw to scale the triangle in which $B=70°$, $C=30°$, $c=10$.
Notice that these parts fix the triangle.
Measure the other parts. (Fig. 31.)

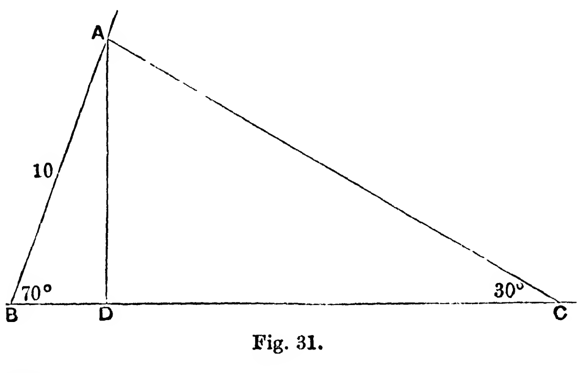


Fig. 31.

2. Taking the triangle of Ex. **1**, what is A?

[Notice that to be given two angles of a triangle amounts to being given three, and the third should be written down at once.]

Draw AD perpendicular to BC. From $\triangle$ADB find AD.

Then from $\triangle$ADC find AC.

[Notice that AD, the connecting link between AB (the side given) and AC

(the side wanted), is the perpendicular through the point A where these two sides meet.]

Draw BE perpendicular to AC, and find BC in an exactly similar way.

Compare your results with what you got in Ex. **1**.

3. In Ex. **2**, BC might be got by finding BD and DC from the first figure. Find it in this way.

4. Find area of $\triangle$ABC by using (i) $\frac{1}{2}$BC . AD, (ii) $\frac{1}{2}$AB . CF, where CF is perpendicular to AB. [See Ex. 60, p. 45.]

5. Go through the steps of Exs. **1** and **2**, taking $B=50°$, $C=53° 8'$, $c=8$. Also find area of the triangle.

6. Draw the triangle in which $B=20°$, $C=150°$, $c=10$.

Measure the other parts.

7. In the triangle of Ex. **6**, what is A? Draw AD perpendicular to BC produced. What is the size of the angle ACD? Find AD and then AC.

Then find BC by drawing BE perpendicular to AC produced.

Also find the area of the triangle.

8. Do the same as in Ex. **7**, taking $B=30°$, $C=126° 52'$, $c=10$.

EXERCISES. XIX.

[Use table, § 40.]

1. Given $B=70°$, $C=14° 29'$, $c=5$, find b.

2. Given $A=30°$, $C=53° 8'$, $c=16$, find a.

3. Given $A=41° 48'$, $B=54°$, $a=8$, solve the triangle and find its area.

4. Given $A=65°$, $B=71°$, $a=10$, find the area.

5. Given $B=70°$, $A=80°$, $c=3$, solve the triangle.

6. Given $A=45°$, $C=75°$, $b=2$, solve the triangle.

7. Given $B=126° 52'$, $C=35°$, $b=10$, find c.

8. Given $A=138° 12'$, $B=11° 32'$, $b=3$, find a.

9. Given $A=18°$, $B=12°$, $c=5$, find the area.

10. Given $B=19° 28'$, $A=100°$, $b=8$, solve the triangle and find its area.

11. Given $A=32°$, $C=110°$, $b=10$, solve the triangle.

12. Given $A=25°$, $B=40°$, $c=8$, solve the triangle.

Case 3. Given three sides.

EXERCISES. XX.

1. Draw to scale the triangle in which

$$a=13, \quad b=14, \quad c=15 \text{ [unit 1 cm.]}.$$

Notice that these parts fix the triangle.

Measure the angles.

2. Without drawing a figure, how can you tell that A and B are both acute angles? Draw CF perpendicular to AB. Does CF fall inside or outside the triangle?

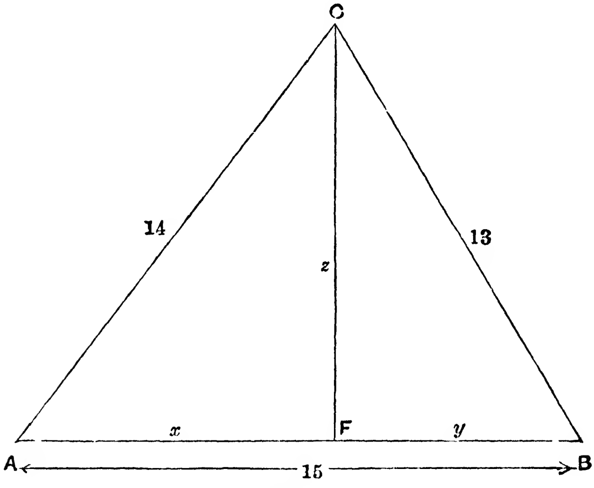


Fig. 32.

Call AF, BF, CF, x, y, z cms. respectively. (Fig. 32.)

What are x^2+z^2, y^2+z^2, x^2-y^2?

What is $x+y$? What is $x-y$?

Hence find x and y.

Which trigonometrical ratios of A and B can you now get?

Find these, and so get A and B. Hence find C.

Compare your results with those obtained in Ex. **1**.

3. Having found x calculate z. [You know x^2+z^2.] Hence get the area of the triangle.

4. Go through the steps of Exs. **1**, **2** and **3**, when $a=2$, $b=3$, $c=4$.

43. We will now work out the example in Exs. XX. 2.

Since c is the greatest side, C is the greatest angle, ∴ A and B are both acute.

∴ CF perpendicular to AB falls inside the triangle.

$$x^2 + z^2 = 14^2,$$
$$y^2 + z^2 = 13^2,$$
$$\therefore\ x^2 - y^2 = 14^2 - 13^2,$$
$$\text{i.e. } (x+y)(x-y) = (14+13)(14-13)$$
$$= 27,$$
$$\text{and } \left.\begin{aligned} x+y &= 15 \\ \therefore\ x-y &= 1{\cdot}8 \end{aligned}\right\},$$
$$\therefore\ \left.\begin{aligned} x &= 8{\cdot}4 \\ y &= 6{\cdot}6 \end{aligned}\right\}.$$

Now
$$\cos \text{A} = \frac{x}{14} = \frac{8{\cdot}4}{14} = {\cdot}6, \qquad \therefore\ \text{A} = 53^\circ\ 08'.$$
$$\cos \text{B} = \frac{y}{13} = \frac{6{\cdot}6}{13} = {\cdot}5077, \quad \therefore\ \text{B} = 59^\circ\ 29'.$$
$$\therefore\ \left.\begin{aligned} \text{A} &= 53^\circ\ 08' \\ \text{B} &= 59^\circ\ 29' \\ \text{C} &= 67^\circ\ 23' \end{aligned}\right\} \text{ Ans.}$$

A good check is to get z from △CFA and also from △CFB.

From △CFA,
$$z = 14 \sin \text{A} = 14 \times {\cdot}8 \qquad = 11{\cdot}2.$$

From △CFB,
$$z = 13 \sin \text{B} = 13 \times {\cdot}8615 = 11{\cdot}1995.$$

We could also get z from
$$z^2 = 14^2 - 8{\cdot}4^2$$
$$= 22{\cdot}4 \times 5{\cdot}6$$
$$= 4 \times 5{\cdot}6 \times 5{\cdot}6,$$
$$\therefore\ z = 2 \times 5{\cdot}6 = 11{\cdot}2.$$

EXERCISES. XXI.

Solve the following triangles and find the area of each. Check by drawing to scale, and by method suggested in § 43.

1. $a=28$, $b=25$, $c=17$.
2. $a=4$, $b=5$, $c=6$.
3. $a=33$, $b=56$, $c=65$.
4. $a=53$, $b=66$, $c=35$.
5. $a=3$, $b=5$, $c=7$.
6. $a=5$, $b=7$, $c=8$.
7. $a=92$, $b=75$, $c=29$.
8. $a=29$, $b=35$, $c=48$.
9. $a=3$, $b=4$, $c=6$.
10. $a=10$, $b=21$, $c=17$.

Case 4. Given two sides and the angle opposite to one of them.

44. It will help you to understand this case if you try to draw the following triangles with instruments, the unit being 1 inch:

(i) $A=30°$, $b=4$, $a=1{\cdot}5$.
(ii) $A=30°$, $b=4$, $a=2$.
(iii) $A=30°$, $b=4$, $a=3$.
(iv) $A=30°$, $b=4$, $a=4$.
(v) $A=30°$, $b=4$, $a=5$.

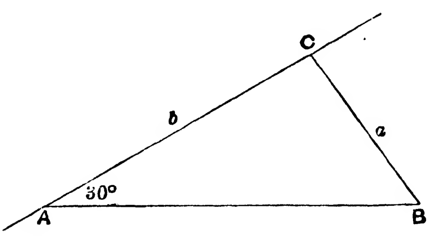


Fig. 33.

First draw freehand any triangle ABC with $A=30°$ and label the sides a and b, so as to see how the given parts lie relatively to each other.

The figure suggests the method of construction.

Draw an angle of 30°, XAY (A).

Along AY measure AC 4 inches long.

B is somewhere along AX and we know its distance from C.

In (i) $CB = 1{\cdot}5''$, (ii) $CB = 2''$, (iii) $CB = 3''$, (iv) $CB = 4''$, (v) $CB = 5''$.

With centre C and radius equal to the given length of CB describe a circle. You will find that

in (i) the circle does not reach AX (Fig. 34);

in (ii) it just touches AX, say at B (Fig. 35);

in (iii) it cuts AX in two points B_1 and B_2, both to the right of A (Fig. 36);

in (iv) it cuts AX in two points, one of which is A. Call the other B (Fig. 37);

in (v) it cuts AX in two points, one to the right of A (B_1), and one to the left of A (B_2) (Fig. 38).

∴ In (i) no triangle can be drawn with the given parts.

In (ii) one triangle can be drawn, viz. ABC, and it will have a right angle at B.

In (iii) two triangles can be drawn, viz. AB_1C and AB_2C.

In (iv) one triangle can be drawn, viz. ABC.

In (v) one triangle can be drawn, viz. AB_1C.

[The point B_2 is the right distance from C but it is on the wrong side of A. In other words the triangle AB_2C has $b = 4$ and $a = 5$, but A is 150° and not 30°.]

In the case, then, in which two sides and the angle opposite to one of them are given, the problem may be impossible, it may have one solution, or it may have two solutions.

When there are *two* solutions we have an instance of what is known as the *Ambiguous Case* in the Solution of Triangles.

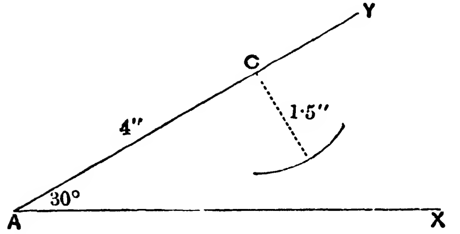


Fig. 34.

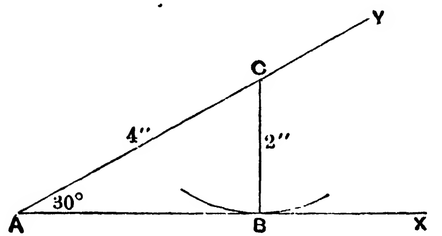


Fig. 35.

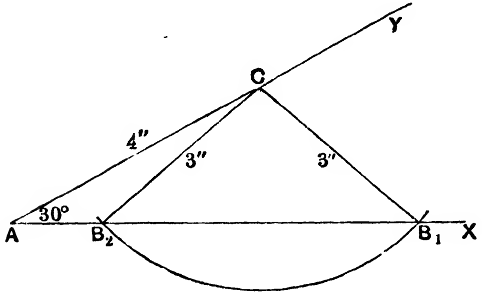


Fig. 36.

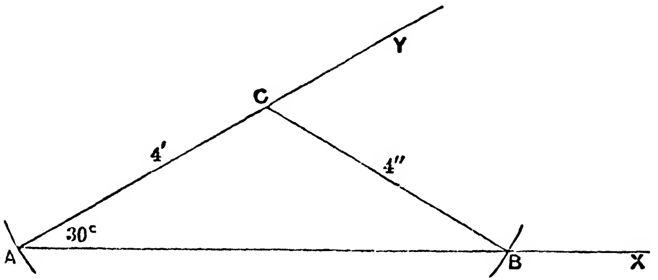


Fig. 37.

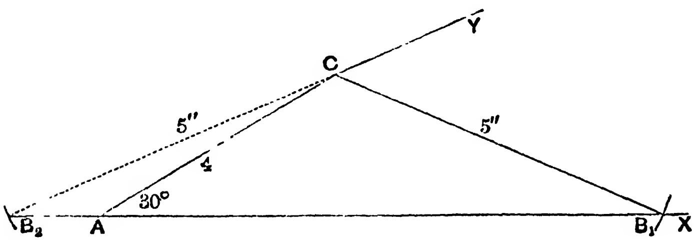


Fig. 38.

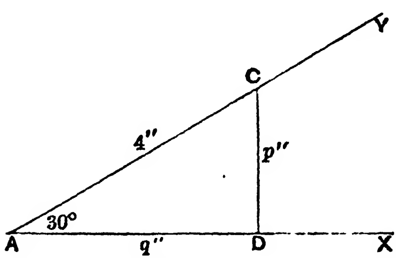


Fig. 39.

45. On reference to your drawings the following points will be at once clear.

Draw CD (p'') perpendicular to AX. (Fig. 39.)

If the given value of $a < p$ the circle described with centre C will not reach AX. This was the case in (i).

If $a = p$ the circle will just touch AX as in (ii).

If $a > p$ the circle will cut AX in two points as in (iii), (iv), and (v).

The two points are on the same side of A, if a is between p and b, as in (iii).

One point coincides with A if $a = b$, as in (iv).

One point is to the right of A if $a > b$, as in (v).

46. These considerations suggest that in solving by Trigonometry we should first find CD.

In each case before putting in the point B, draw a figure showing the angle XAY (= A), AC, and CD perpendicular to AX. (Fig. 39.)

Calculate CD and AD (call them p and q units).

$$\begin{aligned} p &= 4 \sin 30^\circ \\ &= 2, \\ q &= 4 \cos 30^\circ \\ &= 3{\cdot}464. \end{aligned}$$

In (i) $a < p$, $\therefore$ no triangle can be constructed.

In (ii) $a = p$, $\therefore$ the circle centre C and radius a touches AX at D, and B is the same as D.

$$\therefore \left.\begin{aligned} c &= 3{\cdot}464 \\ \text{B} &= 90^\circ \\ \text{C} &= 60^\circ \end{aligned}\right\}.$$

In (iii) $a > p$ and $< b$, $\therefore$ the circle centre C and radius a cuts AX in two points B_1 and B_2 on the same side of A, and each of the triangles AB_1C and AB_2C satisfies the given conditions. (Fig. 40.)

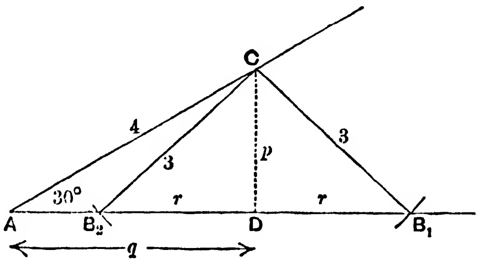


Fig. 40.

Now CB_1B_2 is an isosceles triangle and CD is perpendicular to B_1B_2.

$$\therefore\ B_1D = B_2D\ (= r) \text{ and } \angle CB_1D = \angle CB_2D.$$

Solve $\triangle CB_1D$, and get

$$CB_1D = 41° 48' = CB_2D,$$
$$r = 2{\cdot}236.$$

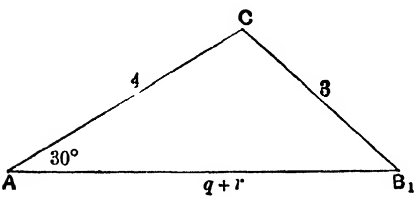


Fig. 41.

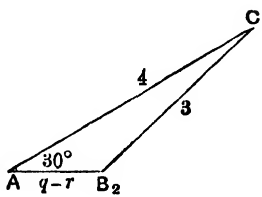


Fig. 42.

$\therefore\ AB_1C = 41° 48'$	$AB_2C = 138° 12'$
$ACB_1 = 108° 12'$	$ACB_2 = 11° 48'$
$AB_1 = q + r = 5{\cdot}70$	$AB_2 = q - r = 1{\cdot}23$;

i.e. we have a double solution, either

$$\left.\begin{array}{l}B = 41^\circ 48' \\ C = 108^\circ 12' \\ c = 5{\cdot}70\end{array}\right\} \text{ or } \left\{\begin{array}{l}B = 138^\circ 12' \\ C = 11^\circ 48' \\ c = 1{\cdot}23.\end{array}\right.$$

In (iv) $a = b$, $\therefore$ the circle cuts AX at A and at another point B. (Fig. 43.)

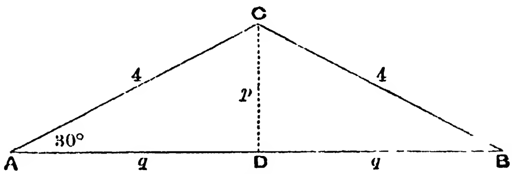


Fig. 43.

There is one triangle ABC which is isosceles, having CA = CB.

$\therefore$ D is the mid-point of AB.

$$\therefore \left.\begin{array}{l}c = 2q = 6{\cdot}93 \\ B = A = 30^\circ \\ C = 120^\circ\end{array}\right\}.$$

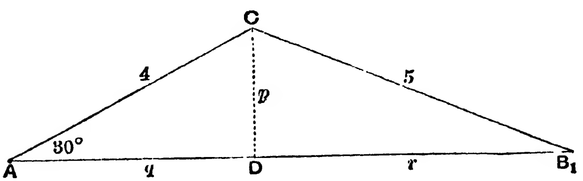


Fig. 44.

In (v) $a > b$, $\therefore$ the circle cuts AX in a point B_1 to the right of A, and in a point B_2 to the left of A. (Fig. 44.)

Only the triangle AB_1C satisfies the conditions.

As in (iii), solve $\triangle CDB_1$,

$$B_1 = 23^\circ\, 35',$$
$$r = 4{\cdot}583,$$
$$\therefore\ c = q + r = 8{\cdot}05.$$
$$\left.\begin{aligned} B &= 23^\circ\, 35' \\ C &= 126^\circ\, 25' \\ c &= 8{\cdot}05 \end{aligned}\right\}.$$

47. In Case (iii) we might find the third side as follows:

As before find p, and hence the two values of B and the two values of C.

Then taking for instance the case in which

$$B = 41^\circ\, 48', \quad C = 108^\circ\, 12',$$

we have now all the angles and two sides and may therefore find the third side by the method of Exs. XVIII., p. 61.

Similarly in Case (v).

EXERCISES. XXII.

Construct and solve these triangles:

1. $A=50^\circ$, $b=5$, $a=3$.	**6.** $B=45^\circ$, $c=4$, $b=3$.
2. $A=50^\circ$, $b=5$, $a=3{\cdot}83$.	**7.** $C=60^\circ$, $b=6$, $c=7$.
3. $A=50^\circ$, $b=5$, $a=4{\cdot}5$.	**8.** $A=150^\circ$, $b=8$, $a=12$.
4. $A=50^\circ$, $b=5$, $a=5$.	**9.** $A=63^\circ$, $c=10$, $a=8{\cdot}91$.
5. $A=50^\circ$, $b=5$, $a=6$.	**10.** $C=35{\cdot}1^\circ$, $a=8$, $c=6$.

Miscellaneous Exercises on Solution of Triangles.

EXERCISES. XXIII.

[Table § 40.]

Solve the following triangles:

1. $B=36^\circ 52'$, $C=60^\circ$, $b=6$.

2. $a=20$, $b=21$, $c=13$.

3.	$b=3$,	$c=4$,	$A=48° 12'$.
4.	$b=3$,	$c=4$,	$C=109° 28'$.
5.	$A=21° 28'$,	$B=20° 20'$,	$c=4$.
6.	$a=4$,	$b=3$,	$B=41° 48'$.
7.	$a=4$,	$b=3$,	$C=120°$.
8.	$a=7$,	$b=9$,	$c=11$.
9.	$a=2$,	$c=6$,	$A=19° 28'$.
10.	$C=30°$,	$A=22° 37'$,	$a=5$.
11.	$a=3$,	$c=5$,	$B=101° 32'$.
12.	$b=3$,	$c=5$,	$B=14° 29'$.

Problems.

48. *Example.* Find the resultant of forces 6 and 8 lbs. wt. acting in directions making an angle of 60° with each other.

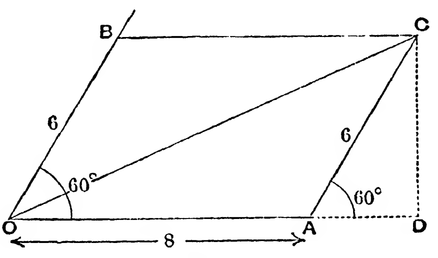


Fig. 45.

Take some length, say 1 cm., to represent a force of 1 lb. wt. and let OA, OB represent the two given forces on this scale, so that

$$OA = 8 \text{ cms.},$$
$$OB = 6 \text{ cms.},$$
$$\angle AOB = 60°.$$

Complete the parallelogram OACB. Then if OC = R cms., the resultant is R lbs. wt. acting along OC.

It is a question, therefore, of solving $\triangle$ OAC in which we know

$$OA = 8 \text{ cms.}, \quad AC = 6 \text{ cms.}, \quad \angle OAC = 120°.$$

Draw CD perpendicular to OA produced.

From $\triangle$ DAC we have

$$AD = AC \cos 60°$$
$$= 3 \text{ cms.},$$
$$DC = AC \sin 60°$$
$$= 5{\cdot}196 \text{ cms.}$$

$\therefore$ in the right-angled triangle ODC,

$$OD = 11 \text{ cms.},$$
$$DC = 5{\cdot}196 \text{ cms.}$$
$$\therefore \tan DOC = \frac{5{\cdot}196}{11} = 0{\cdot}4724.$$
$$\therefore DOC = 25°\ 17',$$

and
$$R = 11 \sec \theta$$
$$= 11 \times 1{\cdot}1059 = 12{\cdot}165.$$

$\therefore$ the resultant is 12·16 lbs. wt. and its direction makes an angle of 25° 17′ with that of the 8 lbs. wt.

EXERCISES.

For problems see Exs. XLI.

CHAPTER VI

SOLUTION OF RIGHT-ANGLED TRIANGLES, USING LOGARITHMS

49. In most cases that are met with in practice, logarithms are used to facilitate the calculation involved in solving right-angled triangles.

The methods used are the same as before, but the long multiplication and division are replaced by addition and subtraction of logarithms.

It is very important to arrange the work in a methodical manner. The fact that logarithms are given to the same number of figures makes this easier, and much of the advantage gained by using logarithms is lost if the arrangement is slovenly.

A few examples are worked out in full to show how the details of the calculation may be arranged.

It must be remembered that logarithms are intended to simplify arithmetic and not to complicate it. For instance it would be foolish to use logarithms in evaluating $5 \sin 40°$.

50. *Example* 1. Solve the triangle in which

$$C = 90°, \ A = 38°, \ c = 96{\cdot}82.$$

(1) $B = 52°$.

(2) $a = c \sin A$

$= 96{\cdot}82 \times {\cdot}6157$

$= 59{\cdot}62.$

No.	log
96·82	1·9860
·6157	$\bar{1}{\cdot}7894$
a	1·7754

(3) $b = c \cos A$

$= 96{\cdot}82 \times {\cdot}7880$

$= 76{\cdot}30.$

No.	log
96·82	1·9860
·7880	$\bar{1}$·8965
b	1·8825

$$\left.\begin{aligned} B &= 52^\circ \\ a &= 59{\cdot}62 \\ b &= 76{\cdot}30 \end{aligned}\right\} \text{ Answer.}$$

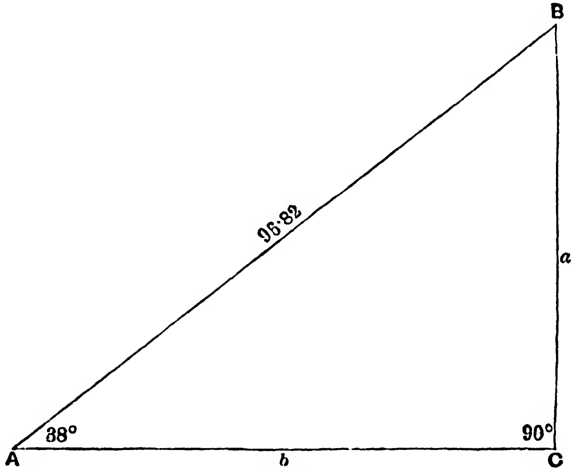


Fig. 46.

Check [see § 32]. Take $a = 59{\cdot}62$, $b = 76{\cdot}30$, $C = 90^\circ$ and find A, B, c.

(1) $\tan A = \dfrac{a}{b} = \dfrac{59{\cdot}62}{76{\cdot}30}$

$= {\cdot}7812.$

$\therefore \quad A = 38^\circ.$

59·62	1·7753
76·30	1·8825
tan A	$\bar{1}$·8928

(2) $\therefore \quad B = 52^\circ.$

(3) $c = b \sec A \left(or \dfrac{b}{\cos A}\right)$

$= 76{\cdot}30 \times 1{\cdot}2690$

$= 96{\cdot}83$;

No.	log
76·30	1·8825
1·2690	0·1035
c	1·9860

or $c^2 = a^2 + b^2$

$= 3554 + 5822$ [sq. table]

$= 9376.$

$\therefore\ c = 96{\cdot}83$ [sq. rt. table].

51. This could also be checked more roughly by drawing to scale; e.g. if we take $\frac{1}{20}$ inch as the unit, AB will be 4·84″ (correct to the nearest hundredth of an inch), BC and AC will be 2·98″ and 3·81″ respectively, i.e. $a = 59{\cdot}6$, $b = 76{\cdot}2$ as nearly as we can tell by drawing.

52. *Example* 2. Solve the triangle in which

$C = 90°,\ c = 64{\cdot}27,\ a = 48{\cdot}93.$

(1) $\sin A = \dfrac{a}{c} = \dfrac{48{\cdot}93}{64{\cdot}27}$

$= {\cdot}7614.$

$\therefore\ \left.\begin{aligned} A &= 49°\ 35' \\ B &= 40°\ 25' \end{aligned}\right\}.$

48·93	1·6896
64·27	1·8080
sin A	$\bar{1}{\cdot}8816$

(2) $^*b^2 = c^2 - a^2$

$= (c + a)(c - a)$

$= 113{\cdot}2 \times 15{\cdot}34.$

$\therefore\ b = 41{\cdot}68$;

113·2	2·0539
15·34	1·1858
b^2	3·2397
b	1·6199

* The method of evaluating $(c^2 - a^2)$ by resolving it into factors is important and opportunities for using it frequently occur in arithmetical calculations.

or $b = a \tan B$
$= 48{\cdot}93 \times {\cdot}8516$
$= 41{\cdot}67$;

or $b = c \sin B$, etc.

No.	log
48·93	1·6896
·8516	$\bar{1}$·9302
b	1·6198

$$\left.\begin{array}{l} A = 49^\circ\ 35' \\ B = 40^\circ\ 25' \\ b = 41{\cdot}67 \end{array}\right\} \text{Answer.}$$

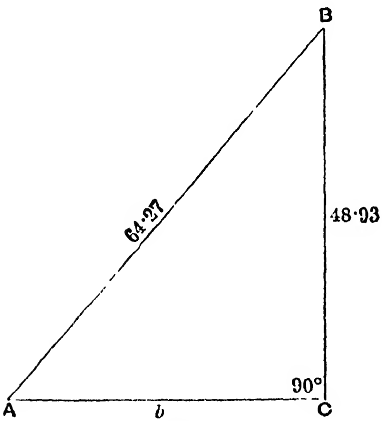


Fig. 47.

Check. Take $A = 49^\circ\ 35'$, $b = 41{\cdot}67$, $C = 90^\circ$, and find the remaining parts.

EXERCISES. XXIV.

Solve triangles in which

1. $C = 90^\circ$, $c = 638$, $B = 71^\circ$.
2. $A = 90^\circ$, $a = 7{\cdot}361$, $B = 25^\circ$.
3. $B = 90^\circ$, $b = 356{\cdot}4$, $C = 55^\circ$.
4. $A = 90^\circ$, $B = 48^\circ$, $b = 961$.
5. $B = 90^\circ$, $C = 80^\circ$, $a = 3{\cdot}142$.
6. $C = 90^\circ$, $A = 32^\circ$, $a = 82{\cdot}76$.
7. $C = 90^\circ$, $a = 1{\cdot}234$, $b = 2{\cdot}345$.
8. $A = 90^\circ$, $b = 68{\cdot}4$, $c = 79{\cdot}5$.
9. $B = 90^\circ$, $a = 7436$, $c = 6291$.
10. $A = 90^\circ$, $a = 78{\cdot}14$, $b = 49{\cdot}2$.
11. $B = 90^\circ$, $b = 639{\cdot}8$, $c = 441{\cdot}2$.
12. $C = 90^\circ$, $c = {\cdot}257$, $a = {\cdot}141$.

"Logarithmic Sines," etc.

53. By using the tables headed "Logarithmic Sines," etc., we can save a few steps in the working of the above examples.

Let us first find out what these tables are.

EXERCISES. XXV.

1. Look out sin 43°. Look out the log of this number.

In the table headed "Logarithmic Sines" look at the entry corresponding to 43°.

Do the same for 35°, 57°, 69°, 81°.

In each case the number in the "Logarithmic Sine" table will be the same as the log of the sine of the angle.

Thus by means of this "Logarithmic Sine*" table we can get the log of the sine of the angle in one step instead of two.

2. Look out tan 43°, then the log of this. Compare with what you find under 43° in the table of "Logarithmic Tangents."

3. Do the same for tangents of 21°, 56°, 71°, 85°.

4. Do the same for cot 37°, cot 48°, sec 19°, sec 67°, cosec 23°, cosec 54°.

54. Examples on the use of these tables.

(1) In a triangle, $C=90°$, $A=38°$, $c=96\cdot82$; find a. [Cp. § 50.]

$$a = c\sin A$$
$$= 96\cdot82 \times \sin 38°$$
$$= 59\cdot61.$$

No.	log
96·82	1·9860
sin 38°	$\bar{1}$·7893
a	1·7753

(2) In a triangle, $C=90°$, $a=59\cdot62$, $b=76\cdot30$; find A. [Cp. § 50.]

$$\tan A = \frac{a}{b} = \frac{59\cdot62}{76\cdot30}.$$

$$\therefore\ A = 38°.$$

59·62	1·7753
76·30	1·8825
tan A	$\bar{1}$·8928

* A better name, which is used in some books, is "Logarithms of Sines." De Morgan, commenting on the absurdity of the name "Logarithmic Sine," said that one might as well call the King of the Country the "Royal Country."

EXERCISES. XXVI.

Solve the following triangles and find their areas:

1.	$C=90^\circ$, $A=50^\circ$, $a=65{\cdot}89$.	**11.**	$C=90^\circ$, $a=837{\cdot}2$, $b=694$.
2.	$A=90^\circ$, $B=62^\circ 18'$, $b=130{\cdot}5$.	**12.**	$A=90^\circ$, $a=726{\cdot}9$, $b=316{\cdot}2$.
3.	$B=90^\circ$, $C=17^\circ 54'$, $c=762{\cdot}3$.	**13.**	$A=90^\circ$, $b=1234$, $c=2345$.
4.	$A=90^\circ$, $B=10^\circ 38'$, $a=27$.	**14.**	$B=90^\circ$, $a=177{\cdot}7$, $c=117{\cdot}7$.
5.	$C=90^\circ$, $B=56^\circ 10'$, $c=963$.	**15.**	$C=90^\circ$, $c=57{\cdot}32$, $a=28{\cdot}58$.
6.	$A=90^\circ$, $C=12^\circ 16'$, $b=73{\cdot}69$.	**16.**	$B=90^\circ$, $b=3579$, $a=1357$.
7.	$C=90^\circ$, $A=54^\circ 18'$, $b=96{\cdot}34$.	**17.**	$B=C=32^\circ 13'$, $a=42{\cdot}81$.
8.	$B=90^\circ$, $A=32^\circ 13'$, $a=16{\cdot}83$.	**18.**	$b=c=32{\cdot}13$, $A=43^\circ 21'$.
9.	$B=90^\circ$, $C=75^\circ$, $a=515$.	**19.**	$a=b=4{\cdot}621$, $A=67^\circ 21'$.
10.	$A=90^\circ$, $B=70^\circ 10'$, $c=11{\cdot}46$.	**20.**	$a=b$, $c=4321$, $C=116^\circ 28'$.

EXERCISES. XXVII.

Log. Sines, etc.

Find the values of

1. $4 \sin 20^\circ \cos 20^\circ \cos 40^\circ$.

2. $4 \cos^3 23^\circ - 3 \cos 23^\circ$.

3. $\dfrac{2 \tan 38^\circ}{1 - \tan^2 38^\circ}$.

4. $\dfrac{\sin^2 21^\circ \times \cos^3 64^\circ}{\tan^2 57^\circ}$.

5. $\dfrac{\sqrt{\cos 40^\circ \times \tan^3 32^\circ}}{\sqrt[3]{\sin 80^\circ \times \cos^2 71^\circ}}$.

6. The horizontal range of a projectile fired in vacuo with a velocity u feet per sec. in a direction making an angle a with the horizontal is $\dfrac{2u^2 \sin a \cos a}{g}$ feet, and the time of flight is $\dfrac{2u \sin a}{g}$ secs., where $g=32{\cdot}2$.

Find the range and time of flight when the velocity of projection is 1200 feet per sec., and the angle of projection (i) 20°, (ii) 40°, (iii) 45°, (iv) 50°, (v) 70°.

7. With the same notation as in the last example the greatest height that the projectile reaches is $\dfrac{u^2 \sin^2 a}{2g}$ feet.

Find the greatest height in each of the same cases as in the last example.

8. Find θ from the equation $\tan \theta = \dfrac{a}{b}$, where $a=29{\cdot}8$, $b=36{\cdot}3$.

Also find ϕ from the equation $\sin \phi = \dfrac{2ab}{a^2+b^2}$ for the same values of a and b.

9. If $\tan A = \dfrac{3\cdot64 \times 7\cdot29}{30\cdot62}$, find $\sin A$.

10. If $\cos A = \dfrac{b^2+c^2-a^2}{2bc}$, find A when $a=8\cdot24$, $b=7\cdot38$, $c=9\cdot76$.

11. If $a^2=b^2+c^2-2bc\cos A$, find a when $b=6\cdot84$, $c=9\cdot78$, $A=50°$.

12. Find A if $\tan\dfrac{A}{2} = \sqrt{\dfrac{(s-b)(s-c)}{s(s-a)}}$, where $a=942$, $b=876$, $c=748$, and s stands for $\dfrac{a+b+c}{2}$.

13. Find θ so that $\tan\theta = \dfrac{b-c}{b+c}\cot\dfrac{A}{2}$, where $b=47\cdot91$, $c=36\cdot42$, $A=78°$.

14. If $a=(b+c)\sin\theta$, where $\cos\theta = \dfrac{2\sqrt{bc}}{b+c}\cos\dfrac{A}{2}$, find a when $c=6\cdot84$, $b=9\cdot78$, $A=50°$.

15. If $a=(b-c)\cos\dfrac{A}{2}\sec\phi$, where $\tan\phi = \dfrac{b+c}{b-c}\tan\dfrac{A}{2}$, find a, when $c=6\cdot84$, $b=9\cdot78$, $A=50°$.

Problems.

55. The ideas involved in the following problems are as a rule no more difficult than those involved in the problems of Exs. XIII. The main difference is that there is more arithmetic and logarithms will be used to simplify the work.

56. *Example* 1.

B is 11·8	miles	N. 35° E.	from	A.
C is 13·7	„	N. 46° W	„	B.
D is 12·3	„	S. 40° W	„	C.

Find the distance and bearing of D from A.

The method we shall adopt is to project the step from A to B on lines running N. and E. respectively, and similarly for the steps from B to C and C to D. By combining the results we shall find the projections of the step from A to D on the N.-S. and E.-W. lines, and from these it is easy to get the required results. [See Exs. X. 10, p. 27.]

Projection of step AB on line running N.

$= \underline{11 \cdot 8 \cos 35°}$

$= 9 \cdot 668.$

No.	log
11·8	1·0719
cos 35°	$\bar{1}$·9134
	0·9853

Projection of step AB on line running E.

$= \underline{11 \cdot 8 \sin 35°}$

$= 6 \cdot 769.$

No.	log
11·8	1·0719
sin 35°	$\bar{1}$·7586
	0·8305

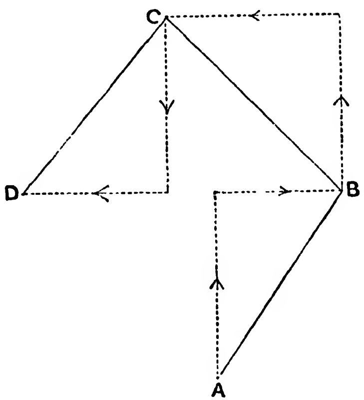


Fig. 48.

Projection of step BC on line running N.

$= \underline{13 \cdot 7 \cos 46°}$

$= 9 \cdot 517.$

No.	log
13·7	1·1367
cos 46°	$\bar{1}$·8418
	0·9785

Projection of step BC on line running W.

$= \underline{13 \cdot 7 \sin 46°}$

$= 9 \cdot 854.$

No.	log
13·7	1·1367
sin 46°	$\bar{1}$·8569
	0·9936

Projection of step CD on line running S.

$= \underline{12 \cdot 3 \cos 40°}$

$= 9 \cdot 423.$

No.	log
12·3	1·0899
cos 40°	$\bar{1}$·8843
	0·9742

Projection of step CD on line running W.

$= \underline{12 \cdot 3 \sin 40°}$

$= 7 \cdot 907.$

No.	log
12·3	1·0899
sin 40°	$\bar{1}$·8081
	0·8980

[Much unnecessary turning over of pages can be prevented by some such plan as this: Begin with a skeleton of the work, consisting of the underlined parts; then fill in all the log sines and log cosines, then the logs of 11·8, 13·7, 12·3, then do the additions and read off the six projections from the page at which the book is open.]

The results may be conveniently collected as follows:

	N.	S.	E.	W.
AB	9·668	—	6·769	—
BC	9·517	—	—	9·854
CD	—	9·423	—	7·907
	19·185	9·423	6·769	17·761
AD	9·762			10·992

In $\triangle$ AMD (Fig. 49) $\tan A = \dfrac{10 \cdot 99}{9 \cdot 762}$,

$\therefore \quad A = 48° \, 23'$,

10·99	1·0410
9·762	0·9895
tan A	0·0515

$$x = 9{\cdot}762 \sec A$$
$$= 14{\cdot}70;$$

No.	log
9·762	0·9895
sec A	0·1778
x	1·1673

i.e. D is 14·7 miles N. 48° 23′ W. from A.

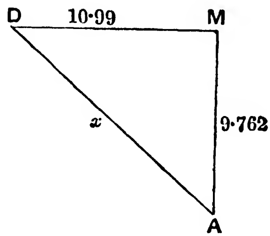


Fig. 49.

Example 2.

PQ is a tower whose height is required. A, B are two stations in the same horizontal plane as Q the foot of the tower. From A, B, the angles of elevation of P are 25° and 20° respectively. If B is 257 feet due E. of A, and the tower is due N. of A, find the height of the tower and its bearing from B.

There are three right-angled triangles as shown in the figure.

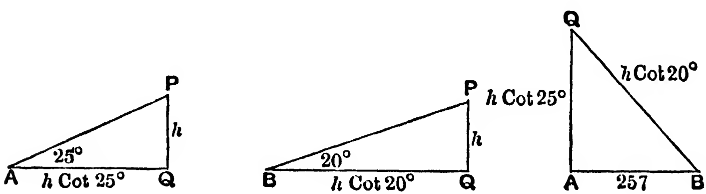


Fig. 50.

Let height of tower be h feet, and let $\angle AQB = \theta$.

Then from the triangles AQP, BQP we can get AQ and BQ in terms of h as in the figure.

From $\triangle$ABQ $\quad \cos\theta = \dfrac{\cot 25°}{\cot 20°}$,

$\therefore \quad \theta = 38° \, 41'$.

No.	log
cot 25°	0·3313
cot 20°	0·4389
cos θ	$\bar{1}$·8924

Also $\quad h \cot 25° = 257 \cot\theta$,

$\therefore \quad h = 257 \cot\theta \tan 25°$

$= 149·7$;

257	2·4099
cot θ	0·0966
tan 25°	$\bar{1}$·6687
h	2·1752

i.e. tower is 150 feet high and bears N. 38° 41′ W. from B.

[h might be got from

$$h^2 (\cot^2 20° - \cot^2 25°) = 257^2,$$

$$\therefore \quad h^2 (2·7475 + 2·1445)(2·7475 - 2·1445) = 257^2,$$

$$\therefore \quad h = \frac{257}{\sqrt{4·892 \times 0·603}}$$

$= 149·6$.

4·892	0·6895
0·603	$\bar{1}$·7803
(Denr.)^2	0·4698
Denr.	0·2349
257	2·4099
h	2·1750]

EXERCISES. XXVIII.

1. The angle of elevation of a tower at a point 138 feet from its base is 38° 24′. Find the height of the tower.

2. A flagstaff is 96·2 feet high. What is the angle of elevation at a point 142 feet from the foot?

3. Find the altitude of the sun when the shadow of a flagstaff 96·2 feet high is 152·3 feet long.

4. Find the length of the shadow of a tower 88·7 feet high when the altitude of the sun is 42° 17′.

5. What will be the altitude of the sun if the shadow of the tower in Ex. **4** is twice as long?

6. B is 26·3 miles E. of A and C is 39·7 miles N. of B. Find the distance and bearing of C from A.

7. A is 12·3 miles S. 71° E. of B. From a ship, A bears N. 44° E. and B bears N. 46° W. Find the distances of A and B from the ship.

8. A is due N. of an observer. He walks 343 yards due W. and its bearing is then N. 29° E. Find his distance from A at each observation.

9. A ship steams 87 miles N. 34° W. How far N., and how far W., will she be from her starting-point?

10. A ship's course is between S. and W. What is it, if after a run of 72·9 miles, she is 48·7 miles W. of the starting-point?

11. In a $\triangle ABC$, $A=90°$, $B=66°$. AD is perpendicular to BC. If $AD=12·3$ cms., find BC.

12. In a $\triangle ABC$, $A=90°$, $B=38° 14'$. AD is perpendicular to BC. If $BC=28·42$ cms., find AD.

13. In a $\triangle ABC$, $A=90°$. AD is perpendicular to BC. If $BC=36·21$ cms. and $AD=14·17$ cms., find AB and AC.

14. O is the centre of a circle, radius 32·91 cms. P is a point such that $OP=76·41$ cms. PT is a tangent to the circle. Find the length of PT and $\angle OPT$.

15. A rhombus circumscribes a circle, radius 3·64″. If an angle of the rhombus be 75°, find the length of its side.

16. Find the resultant of two forces acting at right angles and the angle it makes with the greater force when the forces are

(i) 11·3 and 14·8 lbs. wt.,

(ii) 209·8 and 176·4 gms. wt.

17. Two forces acting at right angles have a resultant 7·84 lbs. wt. One of the forces makes an angle of 29° 13′ with the resultant. Find the forces.

18. Two forces act at right angles. The smaller is 37·94 gms. wt. and makes an angle of 67° 23′ with the resultant. Find the other force and the resultant.

19. Find the resultant of two equal forces, each 317 gms. wt., acting at an angle of 116°.

20. Two equal forces, each 293·6 gms. wt., have a resultant 428 gms. wt. Find the angle between the forces.

21. From a point in the horizontal plane through the foot of a tower a man finds the angle of elevation to be 23°. He walks 268·1 feet towards the tower and finds the angle of elevation to be 46°. Find the height of the tower.

22. A tower is 107·6 feet high. From a certain point in the horizontal plane through the foot of the tower a man finds the angle of elevation of the top to be 32°. How far must he walk towards the tower to make the angle of elevation 64°?

23. A lighthouse is 139 feet high. A rock is 280 yards from it. If a ship steams so that the angle of elevation of the lighthouse is 6° 24′, by how much will she clear the rock?

24. A ship runs 12·5 miles N. 32° W., then 15·3 miles N. 58° E., then 18·1 miles due E. Find her distance and bearing from the starting-point.

25. Same as Ex. **24**, the runs being 10·5 m. N., 15·7 m. W., 5·3 m. N. 68° 30′ W.

26. Same with 8·7 m. N. 28° E., 17 m. N. 70° E., 11·8 m. S. 23° E.

27. From the top of a tower 103 feet high, the angles of depression of the top and bottom of a house are 29° and 46°. Find the height of the house and its distance from the tower.

28. From the top of a cliff known to be 217 feet above sea level at low tide, the angle of depression of a buoy is found to be 11° 17′ at low tide and 9° 57′ at high tide. Find the rise of the tide.

29. An observer on the top of a hill sees two milestones on a straight road which runs directly away from him. Their angles of depression are 28° 12′ and 12° 28′. What is the height of the hill in feet?

30. A traveller in Switzerland sees the tops A, B, C of three peaks, the angle of elevation of C being the greatest. He recognises A and B and finds from the map that

A is 2190 metres distant and 203 metres above him,

B is 3380 .... 364 ....

He suspects that C is the Finisteraarhorn 16,500 metres distant and 1509 metres above him. Does this supposition agree with the figures?

31. In a quadrilateral ABCD, AB = 12·3 cms., BC = 23·4 cms. and DA = 34·5 cms. Also the angles ABC and ACD are right angles. Find the fourth side and the remaining three angles of the quadrilateral.

32. Find the area of a parallelogram whose sides are 13·81 and 16·48 cms., one of the angles being 48°.

33. Find the area of a parallelogram whose diagonals are 7·85 and 10·31 inches, and the acute angle between them 53°.

34. ABCD is a trapezium in which AB is parallel to CD. If $\angle A = 40°$, $\angle B = 63°$, $CD = 3·63''$ and distance between AB and $CD = 1·29''$, find the area of the trapezium.

35. Find the diameter of the circle circumscribing a triangle whose base is 12·63 cms. and vertical angle 53°.

36. Two circles whose radii are 9·85 cms. and 5·42 cms. intersect in two points whose distance apart is 3·85 cms. Find (i) the angle between the radii to a point of intersection, and (ii) the distance between the centres of the circles.

37. AB is the diameter of a semi-circle; P, Q two points on the circumference such that $\angle BAP = 50°$, $\angle BAQ = 35°$; PM, QN are perpendicular to AB. If $AB = 19·35$ cms., find PM, QN, AM, AN. Hence find PQ Also find PQ from $\triangle OPQ$, where O is the centre.

38. A weight is supported by two ropes inclined at angles 38° and 52° to the vertical, and fastened to two points in a horizontal beam 5 feet apart. Find the lengths of the ropes and the distance of the weight below the beam.

39. If the angles in the last example are 43° and 62°, find how far the weight is below the beam.

40. The co-ordinates of two points are

(i) (13·3, 10·9) (28·5, 22·8),

(ii) (−1·82, 4·26) (7·48, −2·91),

(iii) (389, −148) (−217, −301).

Find in each case the length of the line joining them and its inclination to OX.

41. The co-ordinates of one end of a line whose length is 33·71 units are (14·8, 9·4). Its inclination to OX is 41°. Find the co-ordinates of the other end. [Two answers.] Also do this for an angle of 139°.

42. A and B are two points in a horizontal plane. A balloon ascends vertically from a point between A and B. After a certain time the angle of elevation of the balloon is observed simultaneously from A and B. The angles are 47° and 55° 30′. If $AB = 400$ yards, find the height of the balloon.

43. A and B are two points in a horizontal plane 4000 yards apart. A balloon ascends vertically from a point C between A and B, and ascends uniformly at 4 miles an hour. After a certain time the angle of elevation of the balloon from A is 41°, 12 minutes later from B it is 43°. Find the height of the balloon at each observation.

44. A wooden post, 6 feet high, stands on the south side of a wall which runs due E. and W. When the altitude of the sun at noon is 58° 11′ the shadow of the top of the post just reaches along as far as the bottom of the wall. A month later at noon the altitude of the sun is 48° 52′. How high up the wall will the shadow reach?

45. A and B are two landmarks, A 1347 yards N. of B. A ship sailing due N. finds that A bears N. 35° W. and B S. 55° W. Two minutes later A bears N. 55° W. and B S. 35° W. Find the ship's speed. [1 nautical mile = 6080 ft.]

46. A weight of 112 lbs. is supported by two equal ropes each 4 feet long attached to two points in a horizontal beam 6 feet apart. Find the pull along each rope.

Also find what the pull would be if the length of each rope were doubled.

47. A ship sails N. 26° W. 6·02 miles, then N. 64° E. 8·76 miles, then S. $60\frac{1}{2}$° E. 7·41 miles. Find its distance and bearing from the starting-point.

48. Four posts are placed at the corners of a square ABCD whose side is 137 feet. A fifth post E is placed between C and D, 83 feet from C.

X and Y are two trees. It is found that ADX, BEX are straight lines as also are BCY and AEY.

Find the distance XY, and if AB points due N., find the bearing of Y from X.

49. From P the bearing of A is N.E. and of B S. 70° E. From Q due E. of P, the bearing of A is N. 25° E. If B is due S. of A, what is the bearing of B from Q?

50. From a ship sailing due E., A bears N. 60° E. and B bears S. 50° E. Later, A bears N. 30° E. and B S. 21° 40′ E. Show that B is almost exactly due S. of A.

Problems in Three Dimensions.

51. A and B are two stations in the same horizontal plane with the foot of a tower PQ. From A the tower bears N.E., and from B it bears N.W. The angle of elevation of the tower from A is 30°. If B is 287 feet due E. of A, find the height of the tower.

52. A is due S. of a tower PQ 127 feet high. B is due E. of A. The angle of elevation of P from A is 48° and from B it is 20°. Find the distance AB.

53. The elevation of the summit of a mountain which is due N. is 15° 23′. From a station 7000 yards due W. of the first, the elevation is 11° 20′. Find the height of the mountain.

54. ABCD is the floor of a rectangular room, AD being 41 feet long and AB 19 feet. A man standing at A observes that a point on the ceiling immediately over D has an elevation of 10°. What would be the elevation of the same point from B? What is the height of the room, if the height of the man's eye above the floor is 5 feet 4 inches?

55. A door 8 feet high and 3 feet wide swings through an angle of 40°. Calculate the angle between the old and new positions of a diagonal.

56. A wall 20 feet high faces S. 59° 55′ E. Find the width of its shadow on the horizontal plane when the sun is due S. at an altitude of 30°.

57. Two chimneys AB and CD are of equal height. A person standing between them in the line AC joining their bases observes that the angle of elevation of the one nearer to him is 62°. After walking 80 feet in a direction at right angles to AC, he observes their elevations to be 44° and 32°. Find their height and distance apart.

58. There are two trees of equal height on the same side of a straight road 30 feet wide. From a point on the other side of the road directly opposite to one tree the angles of elevation of the tops of the trees are 50° and 20°. Find the height of the trees and the distance between them.

59. O is a point on the diagonal BD of the floor of a room at which the side AB subtends a right angle. From O the angles of elevation of the corners of the ceiling above B and D respectively are 42° and 28°. Find the dimensions of the room, given that the length of the diagonal of the floor is 49 feet.

60. From the top of a cliff 500 feet high a man observes two ships bearing S.E. and S.W., whose angles of depression are 8° and 14°. Find the distance between the ships.

61. At noon, when the sun is due S., an observer sees an aeroplane due W. at an elevation of 23°, and sees the shadow of the aeroplane on a spot due N.W. which he knows to be $1\frac{3}{4}$ miles distant. What is the height of the aeroplane according to these observations?

62. A is due E. of a tower PQ. From B, A bears N. and Q bears N. 35° W. If AB = 460 feet and the angle of elevation of P from A is 25°, find the height of the tower and the angle of elevation of P from B.

63. A spectator on one side of a straight road is looking in the direction perpendicular to the direction of the road, and observes that a telegraph post is exactly opposite and that 40° is the angular elevation of the top of this post. He then turns and observes that 7° is the angular elevation of the top of the next post. Assuming that the height of each post is the same and that they are 80 yards apart, find the height.

CHAPTER VII

SOLUTION OF TRIANGLES, NOT RIGHT-ANGLED, USING LOGARITHMS

57. The methods are those of Chapter V, but multiplication and division are performed by means of logarithms.

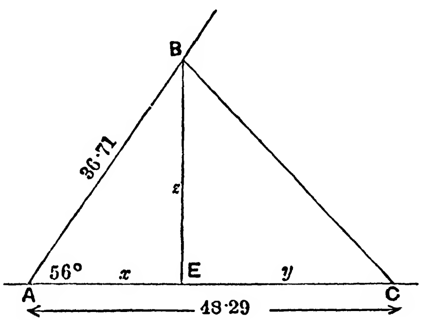


Fig. 51.

58. [Cp. § 41.] *Example.* Solve the triangle in which

$$b = 48{\cdot}29, \quad c = 36{\cdot}71, \quad A = 56°.$$

Draw BE perpendicular to AC. (Fig. 51.)

(1) From $\triangle$BEA $z = 36{\cdot}71 \sin 56°$

No.	log
36·71	1·5648
sin 56°	$\bar{1}$·9186
z	1·4834

(2) $$x = 36{\cdot}71 \cos 56^\circ$$
$$= 20{\cdot}53.$$

No.	log
36·71	1·5648
cos 56°	$\bar{1}$·7476
x	1·3124

(3) $$y = 48{\cdot}29 - 20{\cdot}53$$
$$= 27{\cdot}76.$$

(4) From $\triangle$ BEC, $\tan \mathrm{C} = \dfrac{z}{y}$,
$$\therefore \quad \mathrm{C} = 47^\circ\, 38'.$$

z	1·4831
y	1·4434
tan C	0·0400

(5) $$a = y \sec \mathrm{C}$$
$$= 41{\cdot}19.$$

y	1·4434
sec C	0·1714
a	1·6148

$$\left.\begin{aligned} a &= 41{\cdot}19 \\ \mathrm{B} &= 76^\circ\, 22' \\ \mathrm{C} &= 47^\circ\, 38' \end{aligned}\right\} \text{ Answer.}$$

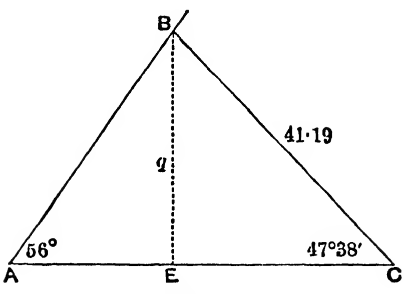


Fig. 52.

Check. Take $a = 41{\cdot}19$, $\mathrm{B} = 76^\circ\, 22'$, $\mathrm{C} = 47^\circ\, 38$ and find b, c, A. (Fig. 52.)

(1) $$\mathrm{A} = 56^\circ\, 00'.$$

(2) Draw BE (q) perpendicular to AC.

$q = a \sin C$
$= 41{\cdot}19 \sin 47° 38'$,
*$c = q \operatorname{cosec} A$
$= 41{\cdot}19 \sin 47° 38' \operatorname{cosec} 56°$
$= 36{\cdot}70$.

No.	log
41·19	1·6148
sin 47° 38′	$\bar{1}$·8685
cosec 56°	0·0814
c	1·5647

(3) Draw CF (r) perpendicular to AB.

$r = a \sin B$
$= 41{\cdot}19 \sin 76° 22'$,
$b = r \operatorname{cosec} A$
$= 41{\cdot}19 \sin 76° 22' \operatorname{cosec} 56°$
$= 48{\cdot}27$.

41·19	1·6148
sin 76° 22′	$\bar{1}$·9875
cosec 56°	0·0814
b	1·6837

[See note to § 56, p. 83.]

EXERCISES. XXIX.

Solve the triangles in which

1.	$a = 36{\cdot}98$,	$b = 42\ 86$,	$C = 74°$.
2.	$b = 4{\cdot}29$,	$c = 3{\cdot}81$,	$A = 111°$.
3.	$a = 247$,	$c = 496$,	$B = 42°$.
4.	$b = {\cdot}1234$,	$c = {\cdot}2345$,	$A = 20° 13'$.
5.	$a = 1001$,	$b = 2002$,	$C = 48° 11'$.
6.	$b = 396{\cdot}1$,	$c = 419{\cdot}3$,	$A = 136° 15'$.
7.	$A = 65°$,	$B = 73°$,	$c = 128{\cdot}4$.
8.	$B = 14°$,	$C = 78°$,	$b = 396{\cdot}1$.
9.	$A = 114°$,	$C = 31°$,	$a = 1{\cdot}236$.
10.	$B = 38° 38'$,	$C = 69° 8'$,	$b = 2345$.
11.	$A = 16° 11'$,	$C = 45° 45'$,	$b = 376$.
12.	$A = 129° 14'$,	$B = 36° 20'$,	$a = {\cdot}428$.

---

* or $c = \dfrac{q}{\sin A}$

$= \dfrac{41{\cdot}19 \sin 47° 38'}{\sin 56°}$

$= 36{\cdot}70$.

41·19	1·6148
sin 47° 38′	$\bar{1}$·8685
Num^r.	1·4833
sin 56°	$\bar{1}$·9186
c	1·5647

13.	$a=6052$,	$b=6765$,	$c=9077$.
14.	$a=47{\cdot}49$,	$b=51{\cdot}38$,	$c=39{\cdot}38$.
15.	$a=8{\cdot}31$,	$b=4{\cdot}23$,	$c=5{\cdot}69$.
16.	$a=205$,	$b=853$,	$c=828$.
17.	$a=346{\cdot}4$,	$b=492{\cdot}6$,	$c=513{\cdot}8$.
18.	$a={\cdot}1357$,	$b={\cdot}3091$,	$c={\cdot}2468$.
19.	$A=42°$,	$b=32{\cdot}68$,	$a=21{\cdot}87$.
20.	$A=42°$,	$b=32{\cdot}68$,	$a=29{\cdot}62$.
21.	$A=42°$,	$b=32{\cdot}68$,	$a=32{\cdot}68$.
22.	$A=42°$,	$b=32{\cdot}68$,	$a=47{\cdot}91$.
23.	$B=55°$,	$c=371$,	$b=304$.
24.	$B=55°$,	$c=371$,	$b=340$.
25.	$B=55°$,	$c=371$,	$b=371$.
26.	$B=55°$,	$c=371$,	$b=529$.
27.	$A=114°$,	$b=2{\cdot}961$,	$a=4{\cdot}736$.
28.	$B=37° 18'$,	$a=94{\cdot}36$,	$b=79{\cdot}23$.
29.	$C=74° 41'$,	$b=821{\cdot}6$,	$c=948{\cdot}7$.
30.	$A=128° 17'$,	$a={\cdot}345$,	$c={\cdot}123$.
31.	$B=54° 23'$,	$b=38{\cdot}97$,	$a=47{\cdot}96$.
32.	$C=78° 16'$,	$c=4763$,	$a=4763$.

For problems see Exs. XLII.

EXERCISES. XXX.

Find areas of triangles in which

1. $a=92{\cdot}63$, $b=87{\cdot}91$, $C=63°$.

2. $b=9{\cdot}142$, $c=8{\cdot}627$, $A=119°$.

3. $b=2468$, $c=4680$, $A=81°$.

4. $b=2468$, $c=4680$, $A=99°$.

5. What is the area of the greatest triangle that can be made with two sides 437 ft. and 562 ft.?

6. Find the area of a parallelogram, whose adjacent sides are 29·43 cms. and 38·92 cms., and included angle 48° 13′.

7. Two sides of a triangle are 987 ft. and 765 ft. Find the included angle if the area is 154,000 sq. ft.

Find areas of triangles in which

8. $A=65°$, $B=73°$, $c=128{\cdot}4$.

9. $A=110°$, $B=23°$, $a=76{\cdot}31$.

10. $a=289$, $b=436$, $c=371$.

11. $A=38°$, $b=4{\cdot}26$, $a=3{\cdot}29$.

CHAPTER VIII

EXERCISES IN GENERALIZING

59. Referring to Exs. III. 1, we see that there are four problems of a precisely similar nature. In each case we are given the distance of the man from the tower and its angle of elevation and want the height. An examination of the solutions will show that what we do each time is to multiply the distance by the tangent of the angle of elevation, and this may be stated symbolically as follows:

If the distance of the man from the tower be d feet and the angle of elevation of the tower be a, its height is $d \tan a$ feet.

60. For a more complicated case, refer to § 33, p. 43. It will be seen that the example worked out there, and the three examples which follow, differ only in the numerical values of the data. In every case we have a figure like Fig. 24, in which we know AB and the angles PAQ, PBQ and have to find PQ.

We go through an exactly similar piece of work every time with these results:

Given that			PQ
AB is	∠PAQ is	∠PBQ is	is found to be
100 ft.	35°	50°	170 ft.
50 yds.	30°	70°	36·5 yds.
5280 ft.	5°	15°	686 ft.
300 yds.	33°	60°	312 yds.

From a mere inspection of this table we could not fill up the fourth column for another set of data, say AB $= 210$ ft., $\angle$ PAQ $= 40°$, $\angle$ PBQ $= 55°$. Yet the number in the fourth column is the result of performing certain operations (the same operations every time) on the given numbers.

If we had a number of problems of the same kind, it would be worth while to state in as simple a form as possible what should be done to the first three quantities to produce the fourth. To do this, go through the same steps as in § 33, taking AB $= a$ ft., $\angle$ PAQ $= \alpha$, $\angle$ PBQ $= \beta$.

We shall have

$$y = x \cot\beta \quad \text{[corresponding to (1) of § 33].}$$

$$a + y = x \cot\alpha \quad \text{[corresponding to (2) of § 33].}$$

$$\therefore \quad a = x(\cot\alpha - \cot\beta),$$

$$\therefore \quad x = \frac{a}{\cot\alpha - \cot\beta} \quad \text{......................(3),}$$

or, in words, PQ is got by dividing AB by the difference of the cotangents of α and β.

(3) is a formula which will give PQ in any case, e.g. in the case proposed above, when

$$AB = 210 \text{ ft.}, \quad \angle PAQ = 40°, \quad \angle PBQ = 55°,$$

PQ will be
$$\frac{210}{\cot 40° - \cot 55°} \text{ ft.}$$

61. Referring to Exs. V. 15, p. 15, we have three problems of the same kind. The required angle in (i) is the angle whose tangent is $\frac{330}{5280}$, and in the general case if the height of the cliff is h ft. and the distance of the boat d ft. the angle of depression is "the angle whose tangent is $\frac{h}{d}$." It is convenient to have a notation for this, and the one usually adopted in English text books is "$\tan^{-1}\frac{h}{d}$."

If x is any positive number we shall take "$\tan^{-1} x$" to mean "**the acute angle whose tangent is x.**" Similarly if x is any positive proper fraction, $\sin^{-1} x$ means "**the acute angle whose sine is x**" and $\cos^{-1} x$ means "**the acute angle whose cosine is x.**"

For instance

$$\tan^{-1} 2 \text{ is } 63^\circ\, 26',$$
$$\sin^{-1} \cdot 8 \text{ is } 53^\circ\, 08',$$
$$\cos^{-1} \cdot 5 \text{ is } 60^\circ.$$

Notice the difference between this notation and that indicated in Ex. 6, p. 28; $\sin^2 A$ means the square of $\sin A$ and is written instead of $(\sin A)^2$. Similarly $(\tan A)^4$ is written $\tan^4 A$; but $\tan^{-1} x$ does not mean $(\tan x)^{-1}$ or $\dfrac{1}{\tan x}$.

You should use this notation in Exs. XXXI. 15, 17, 20, 21, 22 following.

EXERCISES. XXXI.

1. From a buoy the angle of elevation of a ship's masthead is α. If the masthead is h ft. above the water, how far away is the buoy?

2. ABC is a triangle in which A is a right angle. AD (p) is perpendicular to BC. Express p in terms of (i) c and B, (ii) b and B, (iii) a and B, (iv) a, b, c.

3. In the last figure DE (q) is perpendicular to AC. Express q in terms of (i) p and B, (ii) a and B.

4. Find the length of a chord which subtends an angle θ at the centre of a circle of radius r.

5. Find the radius of a circle if a chord of length l subtends an angle θ at the centre.

6. A regular polygon of n sides is inscribed in a circle of radius r. What is the angle subtended at the centre by a side of the polygon? What is the perimeter of the polygon?

7. The same as Ex. 6 for a regular polygon of n sides circumscribed to a circle of radius r.

8. A and B are two stations in a horizontal line with the foot of a tower h ft. high. The angles of elevation of the top of the tower from A and B are α and β respectively ($\alpha < \beta$). Find the distance AB.

9. A wall h ft. high stands a ft. from the side of a house. A ladder with one end on the ground and the other on the side of the house just clears the wall. If the ladder makes an angle θ with the ground, find its length.

10. The tangents to a circle of radius r from an external point contain an angle θ. Find their lengths and the distance of the point from the centre.

11. O is a corner of a rectangular box. The edges OA, OB, OC contain respectively a, b, c units of length. P is the corner opposite to O. What is the length of OP? If OP makes angles α, β, γ with OA, OB, OC respectively, what are $\cos\alpha$, $\cos\beta$, $\cos\gamma$? Show that

$$\cos^2\alpha+\cos^2\beta+\cos^2\gamma=1.$$

12. ABC is an acute-angled triangle. AD (p) is perpendicular to BC. Express

(i) p in terms of c and B.
(ii) b .... p and C.
(iii) b .... c, B, C.

13. In the figure of Ex. **12**, express

(i) BD in terms of c and B.
(ii) DC .... a, c, B.
(iii) b^2 .... a, c, B.

Remembering that $\sin^2 B+\cos^2 B=1$,
you should get $b^2=a^2+c^2-2ac\cos B$.

14. Q is the foot of a tower; A, B are two stations in the same horizontal plane as Q.

B is l ft. due E. of A.

Q is due N. of A.

The angles of elevation of the top of the tower from A, B are α, β respectively.

If h ft. is the height of the tower, express

(i) AQ in terms of h, α.
(ii) BQ .... h, β.
(iii) h .... l, α, β.

15. Two forces P, Q lbs. wt. act at an acute angle α. Find the angle that the resultant makes with the direction of P. [Say $\tan^{-1}$ something.]

16. The base angles of a triangle are α, β (both acute), and its altitude is h ft. What is its area?

17. In a simple horizontal engine, the lengths of the connecting rod and crank are l ft. and r ft. respectively. Find the angle that the crank makes with the line of centres at half stroke.

18. A man at A finds the angle of elevation of the top of a tower PQ to be α. He comes l ft. nearer to B, and finds the elevation to be β. If the two stations and the foot of the tower are in a horizontal line, find the height of the tower. [First find BP as in Exs. XVIII. 2, p. 61.]

19. What is the area of an equilateral triangle of side a inches?

20. Forces P, Q, R lbs. wt. act at a point O in a plane in directions making acute angles α, β, γ respectively with a line OX, on the same side. Write down the resolved parts of each force along OX and perpendicular to OX. What is the resultant of the forces and what is the angle that its line of action makes with OX?

21. What is the vertical angle of a cone which can be formed from a circular sector whose radius is l and arc a?

22. What is the angle between the tangents drawn to a circle of radius r from a point whose distance from the centre is d?

CHAPTER IX

TWO FUNDAMENTAL FORMULAE CONNECTING THE SIDES AND ANGLES OF A TRIANGLE

62. Look at Exs. XVIII. and XIX. of Chapter V. Each time we were given all the angles of a triangle and a side, and had to find the other sides. The process employed was the same on each occasion, and the question naturally arises: Cannot we express in a simple form the result of going through this process, in other words, can we get a formula which shall give us the desired result without the intermediate working?

63. Referring to Exs. XVIII. 1, 2, p. 61, in which we were given

$$\text{B} = 70°,\ \text{C} = 30°\ (\therefore\ \text{A} = 80°),\ c = 10,$$

and had to find b, what we did was this:

From $\triangle$ADB (Fig. 31), $p = 10 \sin 70°$
$= 9{\cdot}397.$

From $\triangle$ADC, $p = b \sin 30°$
$= b \times {\cdot}5.$

$$\therefore\ b \times {\cdot}5 = 9{\cdot}397,$$

$$\therefore\ b = 18{\cdot}79.$$

In Ex. 7, where C is obtuse [$\text{B} = 20°$, $\text{C} = 150°$, $c = 10$],

$$p = 10 \sin 20°$$
$$= 3{\cdot}420,$$

and $p = b \sin 30°$ (the supp. of 150°)
$= b \times {\cdot}5,$ etc.

We could go through the same steps, whatever the values of A, B, C, c.

Two of the angles must necessarily be acute. Suppose A and B are always acute. If C is also acute, we have a figure like Fig. 53.

From $\triangle$ADB, $\qquad p = c \sin B$,

From $\triangle$ADC, $\qquad p = b \sin C$.

$$\therefore \quad b \sin C = c \sin B,$$

$$\therefore \quad \frac{b}{\sin B} = \frac{c}{\sin C}.$$

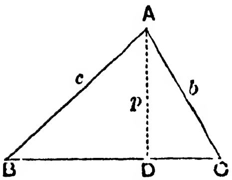


Fig. 53.

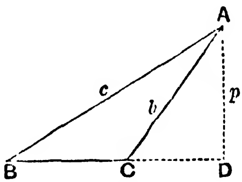


Fig. 54.

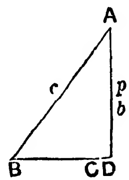


Fig. 55.

If C is obtuse, as in Fig. 54,

$$p = c \sin B,$$

and $\qquad p = b \sin C'$ (where C′ is the supp. of C),

$$\therefore \quad \frac{b}{\sin B} = \frac{c}{\sin C'}.$$

If C is right, as in Fig. 55,

$$b = c \sin B,$$

$$\therefore \quad \frac{b}{\sin B} = \frac{c}{1},$$

and in each case by drawing CF perpendicular to AB we get

$$\frac{a}{\sin A} = \frac{b}{\sin B} \text{ (as in the first case).}$$

$$\therefore \quad \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \text{ (if all the angles are acute),}$$

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin (\text{supp. of } C)} \text{ (if C is obtuse)},$$

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{1} \text{ (if C is right)}.$$

64. Again, referring to Exs. XVI. 1, 2, p. 58, in which we were given $a = 12$, $b = 10$, $C = 36° 52'$ and had to find c,

From $\triangle ADC$ (Fig. 30) we had

$$p = 10 \sin 36° 52' = 6,$$
$$z = 10 \cos 36° 52' = 8,$$
$$\therefore \; y = 12 - 8 = 4;$$

and from $\triangle ADB$, $\quad c^2 = 6^2 + 4^2 = 52,$

$$\therefore \; c = 7{\cdot}211.$$

In Ex. 7, where C is obtuse (= 126° 52′),

From $\triangle ADC$, $\quad p = 10 \sin 53° 08'$ (supp. of 126° 52′)

$$= 8,$$
$$z = 10 \cos 53° 08'$$
$$= 6,$$
$$\therefore \; y = 12 + 6 = 18;$$

and from $\triangle ADB$, $\quad c^2 = 8^2 + 18^2 = 388,$

$$\therefore \; c = 19{\cdot}70.$$

Now let ABC be any triangle in which C is acute (Figs. 56, 57).

From $\triangle ADC$, $\quad p = b \sin C,$

$$z = b \cos C.$$

$\therefore$ in Fig. 56 $\quad y = a - b \cos C,$

and in Fig. 57 $\quad y = b \cos C - a.$

In both figures $\quad c^2 = y^2 + p^2$

$$= (a - b \cos C)^2 + b^2 \sin^2 C$$

[for $(a - b \cos C)^2 = (b \cos C - a)^2$]

$$= a^2 - 2ab \cos C + b^2 (\cos^2 C + \sin^2 C)$$

$$= a^2 + b^2 - 2ab \cos C$$ [see Exs. XI. 8, p. 28].

If C is obtuse (Fig. 58),

$$p = b \sin C',$$
$$z = b \cos C',$$
$$y = a + b \cos C',$$

and
$$c^2 = (a + b \cos C')^2 + b^2 \sin^2 C'$$
$$= a^2 + b^2 + 2ab \cos C'.$$

If C is right, $c^2 = a^2 + b^2$.

$$\therefore \begin{cases} c^2 = a^2 + b^2 - 2ab \cos C \text{ (if C is acute)}, \\ c^2 = a^2 + b^2 + 2ab \cos (\text{supp. of C}) \text{ (if C is obtuse)}, \\ c^2 = a^2 + b^2 \text{ (if C is right)}. \end{cases}$$

E.g. in Exs. XVI. 7, using the second formula, we have

$$c^2 = 12^2 + 10^2 + 2 \times 12 \times 10 \times \cos 53° 8'$$
$$= 144 + 100 + 144$$
$$= 388, \text{ as before.}$$

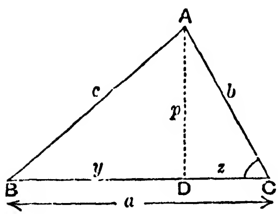


Fig. 56.

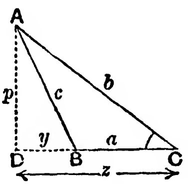


Fig. 57.

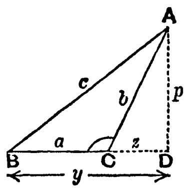


Fig. 58.

65. It would be much more convenient if instead of the three formulae of § 63 we had one which would apply to all cases, and a similar remark may be made of the formulae of § 64.

Now, so far, we have defined and used the terms "sine" and "cosine" in connection with acute angles only.

If the first formula in § 63 applies in all cases, we shall have to take "sine of an obtuse angle" to mean the sine of its supplement, and "sine of a right angle" to mean 1. Let us agree to do this.

Then we can say of all triangles that

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

Similarly we can say of all triangles

$$a^2 = b^2 + c^2 - 2bc \cos A$$

if we agree to take "**cosine of an obtuse angle**" to mean "**− cosine of its supplement,**" and **cosine of a right angle** to mean **0.**

Conversely, if in using these formulae we get $\sin A = \frac{1}{2}$, A may be 30° or 150° (its supplement).

If we get $\cos A = -\frac{1}{2}$, A is the supplement of 60° whose cosine is $\frac{1}{2}$ and so on.

Another way of dealing with obtuse angles which leads to the same results will be found in §§ 66—69.

66. We shall now restate our original definitions of sine and cosine of an angle in a slightly different form. They will agree with the old definitions when applied to acute angles, but will give a meaning to the terms sine and cosine of an obtuse angle.

Let AOB be the angle in question and let P be any point in the arm OB, and OM the projection of OP on OA (Fig. 59). This projection is to be considered positive or negative according as OM points in the same direction as OA or in the opposite direction.

The ratio of this projection to OP, i.e. $\frac{OM}{OP}$ (regard being had to the sign), is called the cosine of AOB.

Thus for an acute angle the cosine is positive, and for an obtuse angle negative.

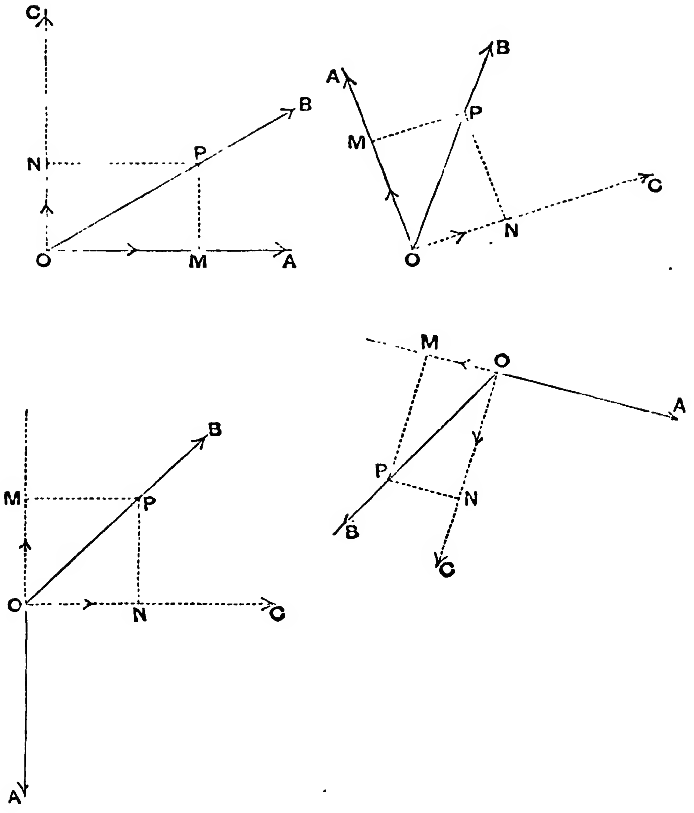


Fig. 59.

EXERCISES. XXXII.

1. If AOB, AOB′ are supplementary angles and OP′ = OP (P′ being in OB′, and P in OB), show that the projections of OP and OP′ on OA are equal in actual length but opposite in sign, and hence that

$$\cos(\text{supp. of A}) = -\cos A.$$

2. If AOB = 90°, what is the projection of OP on OA? What is cos 90° according to our definition? [See Exs. XI. 19, p. 29.]

3. Write down

$$\cos 140°, \cos 123° 48', \cos 100°, \cos 165° 12'.$$

67. Let OC be the position into which OA would come after rotating through a right angle in the same direction in which it rotates through the acute or obtuse angle in question to bring it into the position OB.

Let ON be the projection of OP on OC.

As before this projection is to be considered positive or negative according as ON points in the same direction as OC or in the opposite direction.

The ratio of this projection to OP, i.e. $\frac{ON}{OP}$, is called the sine of $\angle$AOB.

Thus the sine is positive whether the angle is acute or obtuse.

EXERCISES. XXXIII.

1. If AOB, AOB′ be supplementary angles, show that the projections of OP and OP′ on OC are the same; hence

$$\sin(\text{supp. of A}) = \sin A.$$

2. What is sin 90° according to this definition? [See Exs. XI. 18.]

3. Write down

$$\sin 120°, \sin 148° 24', \sin 150°, \sin 167° 18'.$$

68. Now refer to the results collected at the end of § 63. Since sin (supp. of C) = sin C and sin 90° = 1, the three formulae may be stated as one, and we may say that in all cases

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C},$$

remembering that the sine of an obtuse angle is to be taken equal to the sine of its supplement and the sine of a right angle equal to 1.

This is the "**sine formula.**"

69. Similarly since cos (supp. of C) $= -\cos$ C and $\cos 90° = 0$, the three results at the end of § 64 may be stated as one, and we may say that in all cases

$$c^2 = a^2 + b^2 - 2ab \cos C,$$

remembering that the cosine of an obtuse angle is to be taken equal to (– the cosine of its supplement) and the cosine of a right angle equal to 0.

This is the "**cosine formula.**"

EXERCISES. XXXIV.

1. Draw any triangle with angles 50°, 60°, 70°.

Measure the sides and work out

$$\frac{a}{\sin A}, \frac{b}{\sin B}, \frac{c}{\sin C}$$

each to two decimal places. [Use logs if you like.]

2. Do the same for triangles with angles

(i) 45°, 60°, 75°; (ii) 30°, 60°, 90°; (iii) 25°, 45°, 110°.

3. Draw a triangle whose sides are 4, 5, 6 cms.

Measure the angles and work out

$$\frac{\sin A}{a}, \frac{\sin B}{b}, \frac{\sin C}{c}$$

each to two decimal places.

4. Do the same for triangles with sides

(i) 6, 8, 9 cms.; (ii) 3, 4, 5 ins.; (iii) 5, 10, 7 cms.

5. Let ABC be a triangle in which A is acute. About the triangle describe a circle and let BD be the diameter through B. Join CD. What do you know about the angles BCD, BDC? Show that $\frac{a}{\sin A} = d$, where d is the diameter. Hence in an acute-angled triangle show that

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = d.$$

6. In the last example, suppose A is obtuse, what is the angle BDC? Again show that $\frac{a}{\sin A} = d$ and

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = d.$$

USE OF SINE FORMULA.

70. The equation $\frac{a}{\sin A} = \frac{b}{\sin B}$ involves four parts of the triangle, two sides a, b and the angles opposite to them. If three of these four things are given, the fourth can be found; i.e. the formula can be used in two cases.

(1) If two angles (A, B) and the side (a) opposite one of them are given, the side (b) opposite the other angle can be found.

(2) If two sides (a, b) and the angle (A) opposite one of them are given, the angle (B) opposite the other side can be found.

EXAMPLES ON THE USE OF THE SINE FORMULA.

71. Given two angles and the side opposite to one, find the side opposite to the other.

Ex. 1. Given $A = 130°$, $B = 20°$, $c = 4$, find a.

We have $C = 30°$,

$$\frac{a}{\sin A} = \frac{c}{\sin C},$$

$$\therefore\ a = \frac{c \sin A}{\sin C} = \frac{4 \times \cdot 7660}{\cdot 5} \quad \text{(remembering that } \sin 130° = \sin 50°\text{)}$$

$$= 6{\cdot}1280.$$

$$a = 6{\cdot}128. \quad \text{Answer.}$$

Ex. 2. Given $A = 38°$, $B = 65°$, $a = 32{\cdot}64$, find c and b.

We have $C = 77°$,

$$\frac{c}{\sin C} = \frac{a}{\sin A},$$

$$\therefore\ c = \frac{a \sin C}{\sin A}$$
$$= \frac{32{\cdot}64 \sin 77^\circ}{\sin 38^\circ}$$
$$= 32{\cdot}64 \sin 77^\circ \operatorname{cosec} 38^\circ$$
$$= 51{\cdot}65.$$

No.	log
32·64	1·5137
sin 77°	$\bar{1}$·9887
cosec 38°	0·2107
c	1·7131

Similarly, from $\dfrac{b}{\sin B} = \dfrac{a}{\sin A}$

get $b = 48{\cdot}05.$

Check. Take $a = 32{\cdot}64$, $b = 48{\cdot}05$, $c = 51{\cdot}65$ and calculate A as in Ex. 2, § 75; or take $b = 48{\cdot}05$, $c = 51{\cdot}65$, $A = 38^\circ$, and calculate a as in Ex. 2, § 74.

EXERCISES. XXXV.

Use this formula in Exs. XIX., p. 62 and XXIX. 7—12, p. 94.

72. Given two sides and the angle opposite to one, find the angle opposite to the other.

Ex. 1. Given $a = 10$, $b = 20$, $A = 25^\circ$, find B.

$$\frac{\sin B}{b} = \frac{\sin A}{a},$$

$$\therefore\ \sin B = \frac{b \sin A}{a} = \frac{20 \times {\cdot}4226}{10}$$
$$= {\cdot}8452,$$
$$\therefore\ B = 57^\circ\, 42' \text{ or } 122^\circ\, 18'$$

(since by our definition $\sin 122^\circ\, 18' = \sin 57^\circ\, 42'$).

Whichever of these two values of B we take, $A + B < 180^\circ$.

$\therefore$ both values are admissible.

That there are two solutions can also be seen by drawing ...re to scale.

Ex. 2. Given $a = 20$, $b = 10$, $A = 25°$, find B.

$$\frac{\sin B}{b} = \frac{\sin A}{a},$$

$$\therefore \sin B = \cdot 2113,$$

$$\therefore B = 12° \, 12' \text{ or } 167° \, 48'.$$

In this case if we take the larger value of B, $A + B > 180°$.

$\therefore$ there is only one solution, viz. $B = 12° \, 12'$.

Ex. 3. Given $a = 34{\cdot}21$, $b = 43{\cdot}86$, $A = 43°$, solve the triangle.

(1) To find B.

$$\sin B = \frac{b \sin A}{a}$$

$$= \frac{43{\cdot}86 \sin 43°}{34{\cdot}21},$$

$$\therefore B = 61° \text{ or } 119°.$$

No.	log
43·86	1·6421
sin 43°	$\bar{1}$·8338
Numr.	1·4759
34·21	1·5341
sin B	$\bar{1}$·9418

Both solutions are admissible, for

(i) each > 43°, (ii) $119° + 43° < 180°$.

(2) To find c.

There will be two cases.

(i) If $B = 61°$, $C = 76°$ (since $A = 43°$).

$\therefore$ we have $B = 61°$, $C = 76°$, $b = 43{\cdot}86$, find c.

This is like Ex. 2, § 71. [$c = 48{\cdot}66$.]

(ii) If $B = 119°$, $C = 18°$ (since $A = 43°$).

$\therefore$ we have $B = 119°$, $C = 18°$, $b = 43{\cdot}86$, find c. [$c = 15{\cdot}50$.]

EXERCISES. XXXVI.

Solve by this method Exs. XXII., p. 72 and XXIX. 27—32, p. 95.

Use of Cosine Formula.

73. We have proved $c^2 = a^2 + b^2 - 2ab \cos C$.

In exactly the same way $a^2 = b^2 + c^2 - 2bc \cos A$,

$$b^2 = c^2 + a^2 - 2ca \cos B.$$

Looking at the first one, we see that it involves four parts of the triangle, viz. the three sides and the angle C.

$\therefore$ any three of these being given, we can find the fourth.

The formula can therefore be used in three cases.

(1) Given two sides (a, b) and the included angle (C), find the third side (c).

(2) Given the three sides (a, b, c), find the angles.

(3) Given two sides (a, c) and the angle opposite to one of them (C), find the third side (b).

74. Given two sides and the included angle, to find the third side.

Ex. 1. Given $a = 11$, $b = 24$, $C = 120°$, find c.

$$c^2 = a^2 + b^2 - 2ab \cos C$$
$$= 121 + 576 - 2 \times 11 \times 24 \times \left(-\frac{1}{2}\right)$$
$$(\because \cos 120° = -\cos 60°)$$
$$= 961,$$
$$\therefore c = 31.$$

Ex. 2. Given $a = 36{\cdot}21$, $c = 30{\cdot}14$, $B = 78°\,10'$, solve the triangle.

First find b. $\quad b^2 = a^2 + c^2 - 2ac \cos B$.

(a^2 and c^2 can be written down from the square table. $2ac \cos B$ must be worked out by logs.)

a^2	1311
c^2	908·4
$a^2 + c^2$	2219·4
$2ac \cos B$	447·6
b^2	1771·8

No.	log
2	0·3010
36·21	1·5588
30·14	1·4792
cos 78° 10′	$\bar{1}$·3119
$2ac \cos B$	2·6509

$$\therefore b = 42{\cdot}09.$$

We have now all the sides and one angle.

We can therefore find each of the remaining angles by the sine formula.

Now since b is the greatest side, $\therefore$ B is the greatest angle. $\therefore$ A and C are both acute.

To find C*.

$$\sin C = \frac{c \sin B}{b}$$

$$= \frac{30{\cdot}14 \sin 78^\circ 10'}{42{\cdot}09},$$

$\therefore$ C $= 44^\circ 29'$, or its supplement,

but we know C is acute,

$$\therefore C = 44^\circ 29',$$
$$\therefore A = 57^\circ 21'.$$

No.	log
30·14	1·4792
sin 78° 10′	$\bar{1}$·9906
Numr.	1·4698
42·09	1·6242
sin A	$\bar{1}$·8456

Check.

$$\frac{a}{\sin A} = \frac{36{\cdot}21}{\sin 57^\circ 21'},$$

36·21	1·5588
sin 57° 21′	$\bar{1}$·9253
$\frac{a}{\sin A}$	1·6335

$$\frac{c}{\sin C} = \frac{30{\cdot}14}{\sin 44^\circ 29'},$$

30·14	1·4792
sin 44° 29′	$\bar{1}$·8456
$\frac{c}{\sin C}$	1·6336

$$\therefore \frac{a}{\sin A} = \frac{c}{\sin C}.$$

EXERCISES. XXXVII.

Solve by this method Exs. XVII., p. 61 and XXIX. 1—6, p. 94.

* In many cases it is not certain at this stage whether the greater of the two remaining angles is acute or obtuse. It is therefore better always to find the smaller, which must be acute.

75. To find the angles of a triangle when the three sides are given.

For this purpose it is better to re-arrange the formulae, thus:

$$a^2 = b^2 + c^2 - 2bc \cos A,$$

$$\therefore \ 2bc \cos A = b^2 + c^2 - a^2,$$

$$\left.\begin{aligned} \therefore \ \cos A &= \frac{b^2 + c^2 - a^2}{2bc} \\ \text{similarly} \qquad \cos B &= \frac{c^2 + a^2 - b^2}{2ca} \\ \text{and} \qquad \cos C &= \frac{a^2 + b^2 - c^2}{2ab} \end{aligned}\right\}.$$

Ex. 1. Find the angles of a triangle whose sides are 2, 3, 4 ins.

Take $a = 2$, $b = 3$, $c = 4$.

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{9 + 16 - 4}{2 \cdot 3 \cdot 4} = \frac{21}{2 \cdot 3 \cdot 4} = \frac{7}{8} = \cdot 8750,$$

$$\therefore \ A = 28° \, 57'.$$

$$\cos B = \frac{c^2 + a^2 - b^2}{2ca} = \frac{16 + 4 - 9}{2 \cdot 4 \cdot 2} = \frac{11}{16} = \cdot 6875,$$

$$\therefore \ B = 46° \, 34'.$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab} = \frac{4 + 9 - 16}{2 \cdot 2 \cdot 3} = -\frac{1}{4} = - \cdot 2500,$$

$\therefore \ C = 104° \, 29'$ (the supplement of $75° \, 31'$).

[*Check.* $A + B + C = 180°$.]

Theoretically we need only calculate two angles by the formula. The third could then be found by subtracting their sum from 180°. It is better to find all three independently and use $A + B + C = 180°$ as a test of accuracy.

Ex. 2. Find A, given $a = 68{\cdot}47$, $b = 92{\cdot}68$, $c = 71{\cdot}42$.

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc},$$

b^2	8589
c^2	5101
$b^2 + c^2$	13690
a^2	4688
$b^2 + c^2 - a^2$	9002

$$\therefore\ \cos A = \frac{9002}{2 \times 92{\cdot}68 \times 71{\cdot}42},$$

$$\therefore\ A = 47^\circ\ 10'.$$

No.	log
2	0·3010
92·68	1·9670
71·42	1·8538
Den^r.	4·1218
9002	3·9543
cos A	$\bar{1}{\cdot}8325$

EXERCISES. XXXVIII.

Solve by this method Exs. XXI., p. 65 and XXIX. 13—18, p. 95.

76. Given two sides and the angle opposite to one of them, to find the third side.

Ex. 1 Given $a = 7$, $b = 8$, $A = 60^\circ$, find c.

$$a^2 = b^2 + c^2 - 2bc \cos A,$$

$$\therefore\ 49 = 64 + c^2 - 2 \times 8c \times \frac{1}{2},$$

$$\therefore\ c^2 - 8c + 15 = 0,$$

$$\therefore\ (c-3)(c-5) = 0,$$

$$\therefore\ c = 3 \text{ or } 5,$$

i.e. two triangles can be constructed with the given parts.

Ex. 2. Given $a=5$, $b=7$, $B=60°$, find c.

$$b^2 = c^2 + a^2 - 2ca\cos B.$$

$$\therefore\ 49 = c^2 + 25 - 2c \times 5 \times \frac{1}{2},$$

$$\therefore\ c^2 - 5c - 24 = 0,$$

$$\therefore\ (c-8)(c+3) = 0,$$

$$\therefore\ c = 8 \ \textit{or}\ -3,$$

and c must be positive, $\therefore\ c=8$,

i.e. in this case there is only one triangle.

Ex. 3. Given $a=34{\cdot}21$, $b=43{\cdot}86$, $A=43°$, find c.

$$a^2 = b^2 + c^2 - 2bc\cos A,$$

$\therefore\ 1171 = 1923 + c^2 - 2c \times m$ (where m stands for $43{\cdot}86 \cos 43°$),

$$\therefore\ c^2 - 2m\,.\,c + 752 = 0,$$

$$\therefore\ (c-m)^2 - (m^2 - 752) = 0,$$

$$\therefore\ (c - 32{\cdot}07)^2 - 277 = 0,$$

$$\therefore\ (c - 32{\cdot}07)^2 - (16{\cdot}64)^2 = 0,$$

$$\therefore\ (c - 48{\cdot}71)(c - 15{\cdot}43) = 0,$$

$$\therefore\ c = 48{\cdot}71 \ \textit{or}\ 15{\cdot}43$$

No.	log
43·86	1·6421
cos 43°	$\bar{1}$·8641
m	1·5062
m^2	3·0124

Compare § 72, Ex. 3.

EXERCISES. XXXIX.

Find by this method the third side in Exs. XXII., p. 72 and XXIX. 27—32, p. 95.

EXERCISES. XL.

Solve the following triangles:

1.	$a=39{\cdot}48$,	$b=47{\cdot}32$,	$c=26{\cdot}98$.
2.	$B=48°\,32'$,	$C=39°\,16'$,	$a=4{\cdot}29$.
3.	$A=78°$,	$a=28{\cdot}63$,	$b=29{\cdot}27$.
4.	$b=286$,	$c=397$,	$A=140°\,8'$.
5.	$B=143°$	$b=9{\cdot}87$,	$c=7{\cdot}89$.

6.	$A=67^\circ 41'$,	$B=76^\circ 14'$,	$a=875$.
7.	$b=7{\cdot}49$,	$a=5{\cdot}41$,	$C=57^\circ 19'$.
8.	$a=6392$,	$b=7846$,	$c=4297$.
9.	$C=41^\circ$,	$b=78{\cdot}19$,	$c=59{\cdot}71$.
10.	$a={\cdot}854$,	$c={\cdot}971$,	$B=68^\circ 10'$.
11.	$a=7{\cdot}623$,	$b=11{\cdot}91$,	$c=8{\cdot}254$.
12.	$A=115^\circ 3'$,	$C=25^\circ 14'$,	$b=286{\cdot}5$.
13.	$b=874{\cdot}0$,	$c=682{\cdot}3$,	$A=38^\circ 41'$.
14.	$b=876$,	$c=795$,	$B=123^\circ$.
15.	$A=117^\circ 48'$,	$B=26^\circ 27'$,	$a=4{\cdot}293$.
16.	$a=44{\cdot}76$,	$b=62{\cdot}45$,	$c=76{\cdot}83$.
17.	$c=92{\cdot}85$,	$a=68{\cdot}41$,	$A=37^\circ 23'$.
18.	$a=54{\cdot}21$,	$b=74{\cdot}16$,	$c=36{\cdot}75$.
19.	$A=28^\circ 19'$,	$C=31^\circ 15'$,	$b=76{\cdot}24$.
20.	$a=238{\cdot}9$,	$b=642{\cdot}8$,	$C=123^\circ 16'$.

21. Prove that the **area of any triangle** $=\frac{1}{2}$ **bc sin A.**

22. Use the formula of Ex. **21** to solve Exs. XXX. 1—7, p. 95.

23. Use the sine formula and the formula of Ex. **21** to solve Exs. XXX. 8—11.

Problems.

77. *Ex.* Find the resultant of forces 6 and 8 lbs. wt. acting in directions making an angle of 60° with each other. [See Ex. and figure at end of Chapter V.]

With the same notation as for Fig. 45.

$$R^2 = 8^2 + 6^2 - 2 \,.\, 8 \,.\, 6 \,.\, \cos 120^\circ$$

$$= 64 + 36 + 2 \,.\, 8 \,.\, 6 \,.\, \frac{1}{2}$$

$$= 148,$$

$$\therefore\ R = 12{\cdot}17,$$

and
$$\sin\theta = \frac{6\sin 120^\circ}{12\cdot 17}$$
$$= \frac{6\sin 60^\circ}{12\cdot 17},$$
$$\therefore\ \theta = 25^\circ\, 16'$$
(since θ is certainly acute).

No.	log
6	0·7782
sin 60°	$\bar{1}$·9375
	0 7157
12·17	1·0854
sin θ	$\bar{1}$·6303

$\therefore$ resultant is 12·17 lbs. wt. acting in a direction making an angle of 25° 16′ with the 8 lbs. wt.

EXERCISES. XLI.

[Use table in § 40 when convenient.]

1. Two ships leave harbour together. One sails due N. at 8 knots, the other N. 60° W. at 10 knots. Find the distance and bearing of the first from the second after 2 hours.

2. Wishing to find the distance of an inaccessible object C, I measure a base line AB of 100 yards and also the angles ABC (80°) and BAC (70°). Find the distance AC.

3. A, B, C are three towns, A lying N. of the line BC. C is 7·5 miles due E. of B. A is 6·5 miles from B and 7 miles from C. Find the bearing of A from B and C.

4. A ship steams 10 miles due E. and then 12 miles N. 30° E. Find her bearing and distance from the starting-point?

5. A sailing ship goes from A to B on two courses. She first sails 10 miles N.E. from A to C and then 5 miles N. 60° W. from C to B. On what course would she have had to sail to go straight from A to B?

6. A ship sails 10 miles S. 36° E. and then 8 miles S. 84° W. Find her distance and bearing from the starting-point.

7. A and B are two landmarks, A being due W. of B and 5 miles distant from it. From a ship A bears N. 28° 16′ E. and B bears N. 65° 8′ E. Find the distances of the ship from A and B.

8. A weight hangs from two hooks 6 feet apart in a horizontal beam by two strings 4 feet and 5 feet long. Find what angles these strings make with the vertical. Also find how far the weight is below the beam.

9. In a simple crane with a vertical post, the tie and jib make angles of 65° and 35° with the upward vertical, and their points of attachment to the post are 4 feet apart. Find the lengths of the tie and jib.

10. A flagstaff stuck in the ground leans over towards the North, and makes an angle of $8° 8'$ with the vertical. The length of the shadow at mid-day when the sun's altitude is $45°$ is 100 feet. Find the length of the flagstaff.

11. A flagstaff 100 ft. long leans over towards the East at an angle of $11° 32'$ to the vertical. What will the angle of elevation of the top of the flagstaff be from a point (i) 80 feet East, (ii) 80 feet West of it?

12. A vertical tower stands at the top of a straight road which slopes at an angle of $5°$ to the horizon. From a point in the road 100 feet from the foot of the tower the angle of elevation of the top is $60°$. Find the height of the tower.

13. OX, OY are two lines at an angle of $60°$. A is a point in OX, 4″ from O. Show that there are two points each of which is equidistant from OX, OY and 3″ from A, and calculate the distance between these two points.

14. The sides of a triangle are 8, 10, 12 cms. Find the length of the median drawn to the mid-point of the longest side.

15. The parallel sides of a trapezium are 2″ and 5″ and the oblique sides 3·5″ and 4·5″. Find the angles.

16. A man at A wishes to find the height of a tower PQ. He finds the angle of elevation to be $36° 52'$. He comes to B 100 feet nearer and finds the angle of elevation to be $66° 52'$. If the two stations and Q the foot of the tower are in a horizontal line, find the height of the tower.

17. In a simple horizontal engine the connecting rod is 3 feet long and the crank 1 foot. Find the angle that the crank makes with the line of dead centres at $\frac{1}{4}$ stroke, $\frac{1}{2}$ stroke, $\frac{3}{4}$ stroke.

18. With the same engine as in Ex. **17**, find the distances of the cross-head from its mean position when the crank makes angles of $30°$, $60°$, $90°$, $120°$ with the line of dead centres.

19. Find the magnitude and direction of the resultant of

(i) Forces 2 and 3 lbs. wt. acting at an angle $60°$.
(ii) .... 4 and 5 .... $36° 52'$.
(iii) .... 5 and 8 .... $120°$.

20. Forces 3, 4, 6 lbs. wt. acting at a point are in equilibrium. Find the angles between their lines of action.

21. In a circle of diameter 10 cms. AB and AC are two chords whose lengths are 7 and 8 cms. respectively. Find BC, when AB and AC are (i) on the same side, (ii) on opposite sides of the diameter through A.

22. A circle is described about a triangle whose sides are 4, 5, 6 cms. Find its radius. (First find an angle of the triangle.)

23. In a convex quadrilateral ABCD, $A=120°$, $AB=6''$, $AD=10''$, $CB=15''$, $CD=13''$. Find the other angles of the quadrilateral.

24. From a ship steaming due N., a lighthouse bears N. 34° W. Half-an-hour later it bears N. 64° W. After how long will it bear S. 58° W.?

25. The legs of a pair of compasses are each 12 cms. long, and the pencil leg has a joint 7 cms. from the hinge. The compasses are used to describe a circle of 12 cms. radius and the pencil leg is bent at the joint so that the pencil is perpendicular to the paper. Find the angle between the legs of the compasses.

Problems in Three Dimensions.

26. From A, due N. of a tower PQ, the angle of elevation of P the top of the tower is 45°. From B, N. 60° E. of the tower, it is 53° 08′. If A, B, Q are in a horizontal plane, and if $AB=100$ feet, find the height of the tower. $\left[\tan 36° 52' = \frac{3}{4}.\right]$

27. From A, due S. of a tower, the angle of elevation of the top is 36° 52′. From B, which is 100 yards N. 80° E. from A, the tower bears N. 70° W. Find the height of the tower.

28. From A a tower known to be 100 ft. high bears due N. and the angle of elevation of the top is 53° 08′. From B the tower bears N. 60° W. and the angle of elevation is 45°. Find the distance and bearing of B from A.

29. The angles of elevation of the top of a tower from three points A, B, C in a horizontal plane are each 60°. Find the height of the tower and its bearing from A, given that B is 100 ft. N.W. from A, and C is 160 ft. N. 15° E. from A.

30. From two points A, B at the same height on a cliff and 500 ft. apart the angles of depression of a buoy are 30° and 36° 52′ and the angle subtended by AB at the buoy is 60°. Find the height of the cliff.

Harder Problems.

78. *Ex.* 1. A man wishes to find the height of an inaccessible tower. He finds the angle of elevation to be 35°. He comes 100 ft. nearer and finds the angle of elevation to be 50°. Find the height of the tower. [See § 33.]

(1) In $\triangle$PAB we know one side (AB = 100′) and all the angles; ∴ we can find PB (call it b').

(2) In $\triangle$PBQ, Q is a right angle, and we know PB and the angles; ∴ we can find PQ (x').

(1) $$\frac{b}{\sin 35^\circ} = \frac{100}{\sin 15^\circ},$$

$$\therefore\ b = 100 \sin 35^\circ \operatorname{cosec} 15^\circ;$$

(2) $$x = b \sin 50^\circ$$
$$= 100 \sin 35^\circ \sin 50^\circ \operatorname{cosec} 15^\circ$$
$$= 169{\cdot}8;$$

No.	log
100	2·0000
sin 35°	$\bar{1}$·7586
sin 50°	$\bar{1}$·8843
cosec 15°	0·5870
	2·2299

i.e. height of tower = 170 feet (to nearest foot).

Ex. 2. To find the distance between two inaccessible objects P and Q, two stations A and B are chosen, A being 200 yards W. from B. From A, P bears N. 8° W. and Q bears N. 65° E. From B, P bears N. 30° W. and Q bears N. 34° E. Find the distance and bearing of Q from P.

From the given bearings, the angles marked on the figure can easily be found (Fig. 60).

We can find PQ (x yds.) from $\triangle$PAQ if we know AP and AQ ($\angle$PAQ being already known).

In $\triangle$PAB we know one side (AB) and the angles; ∴ we can get AP (p yds.).

In $\triangle$QAB we know one side (AB) and the angles; ∴ we can get AQ (q yds.).

(1) In $\triangle$PAB

$$\frac{p}{\sin 60^\circ} = \frac{200}{\sin 22^\circ},$$

$$\therefore\ p = 200 \sin 60^\circ \operatorname{cosec} 22^\circ.$$

200	2·3010
sin 60°	$\bar{1}$·9375
cosec 22°	0·4264
p	2·6649
p^2	5·3298

(2) In $\triangle QAB$

$$\frac{q}{\sin 124^\circ} = \frac{200}{\sin 31^\circ},$$

$$\therefore \quad q = 200 \sin 56^\circ \operatorname{cosec} 31^\circ.$$

No.	log
200	2·3010
sin 56°	$\bar{1}$·9186
cosec 31°	0 2882
q	2·5078
q^2	5·0156

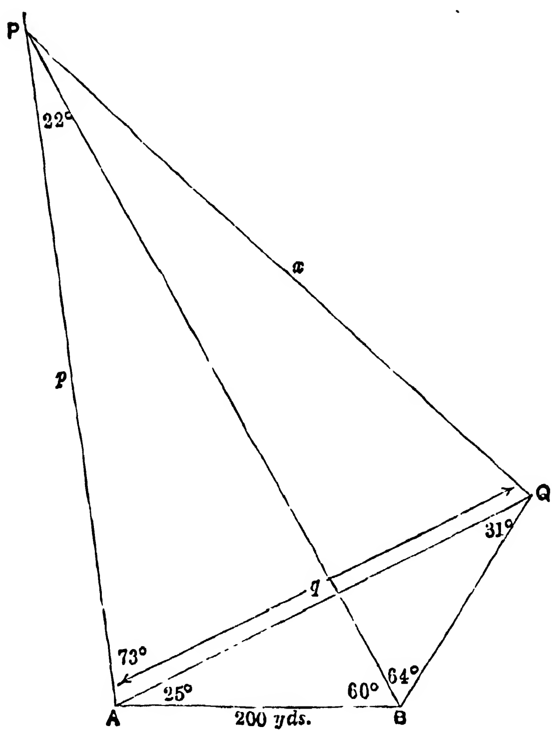


Fig. 60.

(3) In $\triangle$PAQ

$$x^2 = p^2 + q^2 - 2pq\cos 73^\circ.$$

$$\begin{aligned} p^2 &= 213700 \\ q^2 &= 103600 \\ \hline p^2+q^2 &= 317300 \\ 2pq\cos 73^\circ &= 87020 \\ \hline x^2 &= 230280 \\ \therefore\ x &= 479{\cdot}9 \end{aligned}$$

No.	log
2	0·3010
p	2·6649
q	2·5078
cos 73°	$\bar{1}$·4659
$2pq$ cos 73°	4·9396

Since the directions of AP and AQ are known, we can get the bearing of Q from P if we know either of the angles APQ or AQP. Let us calculate the smaller one. [See Note on Ex. 2, § 74.]

$$\sin \text{APQ} = \frac{q \sin 73^\circ}{x}$$

and APQ is acute.

$$\therefore\ \text{APQ} = 39^\circ\ 54'$$

and PA runs S. 8° E.

$\therefore$ bearing of Q from P is S. 47° 54′ E.

So that Q is 480 yds. S. 47° 54′ E. from P.

q	2·5078
sin 73°	$\bar{1}$·9806
q sin 73°	2·4884
x	2·6811
sin APQ	$\bar{1}$·8073

[A good check can be got by finding BP from $\triangle$ABP, BQ from $\triangle$ABQ and then solving $\triangle$BPQ.]

Ex. 3. A and B are two positions 750 feet apart in the same horizontal plane. From A the angle of elevation of the top of a tower is 15°, from B it is 20°. Further the tower is due North of B and N. 50° E. from A. Find the height of the top of the tower above the horizontal plane through A and B.

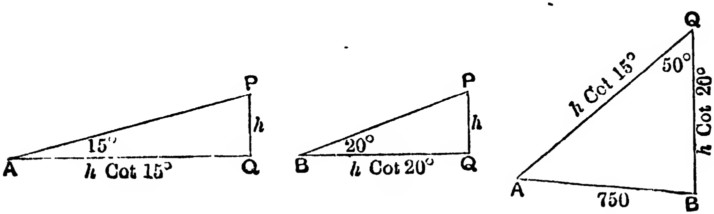


Fig. 61.

P is the top of the tower and Q its foot (Fig. 61). Let height of tower $= h$ feet and let AQ and BQ be a and b feet respectively.

From $\triangle$AQP $\quad a = h \cot 15° = h \times 3{\cdot}7321.$

" $\quad\triangle$BQP $\quad b = h \cot 20° = h \times 2{\cdot}7475.$

" $\quad\triangle$ABQ $\quad AB^2 = QA^2 + QB^2 - 2QA \,.\, QB \cos 50°.$

$\therefore\ 750^2 = h^2\,[\cot^2 15° + \cot^2 20° - 2 \cot 15° \cot 20° \cos 50°].$

$$\cot^2 15° = 13{\cdot}93$$
$$\cot^2 20° = 7{\cdot}549$$
$$\cot^2 15° + \cot^2 20° = 21{\cdot}479$$
$$2 \cot 15° \cot 20° \cos 50° = 13{\cdot}18$$
$$\therefore\ 750^2 = h^2 \times 8{\cdot}30,$$
$$\therefore\ h = \frac{750}{\sqrt{8{\cdot}30}}$$
$$= 260.$$

$\therefore$ height of tower $= 260$ ft.

No.	log
log 2	0·3010
cot 15°	0·5719
cot 20°	0·4389
cos 50°	$\bar{1}$·8081
	1·1199
750	2·8751
* $\sqrt{8{\cdot}30}$	0·4596
	2·4155

* [$\log \sqrt{8{\cdot}30} = \frac{1}{2} \log 8{\cdot}30$.]

Note. See p. 46 for a hint on the making of a model.

EXERCISES. XLII.

1. A lighthouse is due W. of a church, and 7 miles from it. From a ship the lighthouse bears N. 10° W., and the church N. 30° E. Find the distance of the ship from each.

2. A flashing light is 10 miles S. 30° E. from a Coastguard station. From a ship the light bore due E., and the angle subtended between the Coastguard station and the flash light was 80°. Find the distance of the ship from the light, and the bearing of the Coastguard station from the ship.

3. Two ships A and B leave harbour together. A sails S. 12° W. at a rate of 10·5 knots, and B S. 67° E. at a rate of 8 knots. What will be their distance apart at the end of 2 hours, and what will be the bearing of A from B?

4. A, B, C are three towns, A being on the N. side of BC. A is 180 miles from B, and 120 miles from C; B is 145 miles N.W. of C. Find the bearings of A from B and C.

5. An object on shore bears from a ship N. 30° W. After the ship has sailed N. 30° E. 6 miles the object bears N.W. Find the distance of the ship from the object at the second observation.

6. Wishing to find my distance from an inaccessible object, C, I measure a base line, AB, of 600 yards; at A, I find the angle BAC to be 39° 40′; at B, I find the angle ABC to be 74° 10′. Find AC.

7. A privateer is at anchor 10 miles S.W. of a harbour; a merchantman leaves it in the direction S. $78\frac{3}{4}$° E. On what course and at what rate must the privateer sail to come up with the merchantman in $1\frac{1}{2}$ hours, if the latter is running at 9 knots?

8. A ladder whose length is 30 feet stands against a wall at an angle of 60° with the horizon; at what distance from its top must another ladder of the same length be fastened so as to reach a window 48 feet from the ground, and what angle will the second ladder make with the horizon?

9. The adjacent sides of a parallelogram are 38·5 and 49·7 feet, and one diagonal is 51·2 feet. Find the other diagonal and the area of the parallelogram.

10. In a simple horizontal engine the connecting rod is 43″ long and the crank 13·5″ long. Find the angle that the crank makes with the line of dead centres at $\frac{1}{4}$ stroke, $\frac{1}{2}$ stroke, $\frac{3}{4}$ stroke.

11. With the same data as in Ex. **10**, find the positions of the crosshead when the crank makes angles of 35°, 90°, 140° with the line of dead centres.

12. Find the magnitude and direction of the resultant of (i) 3·28 lbs. wt. and 4·63 lbs. wt. at an angle of 48°, (ii) 568 gms. wt. and 749 gms. wt. at an angle of 146°.

13. Three forces, 2·96, 3·84, 4·26 lbs. wt., acting at a point are in equilibrium. What are the angles between their directions?

14. A is the most southerly point on the rim of a circular pond, 1530 yards in diameter. B and C are two other points on the rim N.E., and N. 20° W. of A. Find the length of BC.

15. Newhaven is 45 miles due E. of Portsmouth. From Newhaven, Dieppe bears S. 35° E. and from Portsmouth it bears S. 55° E. Find the distance from Newhaven to Dieppe.

16. A lighthouse is at a distance of 4·5 miles from a battleship in a direction N. 34° E. Find the bearing and distance of the lighthouse from a second battleship which lies 6 miles S. 22° E. from the first?

17. A and B are two objects on opposite sides of a wood. C is a third station from which both A and B are visible. CA = 762 yards, CB = 1021 yards and the angle subtended at C by AB is 114°. Find the distance AB.

18. The angle of elevation of the top of a tower from A is 37°. From B, 247 feet nearer, the angle of elevation is 67°. Find the height of the tower.

19. A and B are two consecutive milestones on a straight road running due N. From A a church tower bears N. 41° W., and from B it bears N. 70° W. How far is the tower to the West of the road?

20. From a barge moving along a straight canal the angle of elevation of a bridge is 10°, 10 minutes later it is 15°. How long will it be before the barge reaches the bridge?

21. A lighthouse stands on the top of a rock. From a boat the angle of elevation of the top of the lighthouse is 18°. The boat is rowed 120 yards directly towards the lighthouse, and the angles of elevation of the top and bottom of the lighthouse are found to be 38° and 26°. Find the height of the lighthouse.

22. From the top and bottom of a tower 100 feet high, the angles of elevation of the top of a hill are 11° and 14° respectively. Find the height of the hill.

23. A is a point in a level plain and the bearing of a certain church spire from A is N. 50° E. Two men set out from A; the first walks 1200 yards due N., and the second a mile due E. If the spire is now in the line joining the positions of the two men, find the distance of the spire from A.

24. A ship is moving due W. at 13 knots. At noon a lighthouse bears N. 58° W. At 12.15 it bears N. 30° W. When will it bear N. 40° E.?

25. A man finds that two towers X and Y subtend an angle of 80° at his eye. He walks 75 yards towards Y and finds that the angle subtended is 146°. Find his distance from X at each observation.

26. A balloon is vertically over a point which lies in a direct line between two observers who are 2000 feet apart, and who note the angles of elevation of the balloon to be 35° 30′ and 61° 20′. Find the height of the balloon.

27. A man standing on the bank of a straight river sees two objects on the further side and the lines joining his position to them make with the direction of flow of the river angles of 51° 36′ and 71° 48′. He walks down stream until the objects are seen in line and finds that the line joining his position to them now makes an angle of 104° 57′ with the direction of flow of the river. He measures the distance he has walked and finds it is 150 yards. Find the distance between the objects.

28. From the point O the three straight lines OA, OB, OC are drawn in the same plane, of lengths 1″, 2″, 3″ respectively and with the angles AOB, BOC each equal to 60°. Find the angle ABC.

29. From a ship the bearings of two objects A and B are N. 30° E. and N. 20° W. After the ship has sailed 5 miles due N., the bearings are N. 60° E. and N. 40° W. Find the distance and bearing of A from B.

30. It is required to find the distance between two distant objects P and Q. A base line XY of 1000 yards is measured, and the angles YXP, YXQ, XYP, XYQ are found to be 95°, 27°, 43°, 105° respectively. Find PQ

31. From a ship steaming N. 43° E. at 12 knots the bearings of A and B are N. 64° E. and N. 20° E. Half-an-hour later A bears due E., and B due N. Find the distance and bearing of A from B.

32. A is 6·8 miles N. 33° W. from B. From A, P bears N. 75° E., and Q bears N. 80° W. From B, P bears N., and Q bears N. 51° W. Find the distance and bearing of Q from P.

33. B is 5·2 miles S. 58° 12′ E. from A. From a ship, A bears N. 38° 40′ E. and B bears N. 57° E. After the ship has steamed for 20 minutes towards A, B bears N. 72° E. Find the speed of the ship.

34. A is 5·3 miles S. 62° E. from B. From a ship, A bears due E. and B due N. The ship is steaming N. 70° E. at a speed of 10 knots. How long will it be before she is on the line joining A and B?

35. B is 12 miles N. 62° E. from A, C is 5·1 miles N. 57° W. from B, and D is 10·4 miles S. 19° E. from C. Find the distance and bearing of D from A.

36. In a semi-circle whose diameter AB is 8·74″ are two chords AC, AD of lengths 6·85″ and 5·21″. Find the angles BAC, BAD, CAD. Hence find the angle subtended at the centre by CD and the length of CD. [Two cases.]

37. ABCD is a cyclic quadrilateral. AB = 2·31″, BC = 3·72″, CD = 1·81″, ∠B = 56°. Find CA, the radius, the angles subtended at the centre by the chords, AB, BC, CD, and the length of DA.

38. A is 10·8 miles N. 65° W. from B. From a ship sailing due N. at 18 knots B bears N. 63° E. 25 minutes later A bears N. 50° E. How far is the ship from A at the second observation?

[Draw AO due S. equal to ship's run. Solve △AOB.]

39. A, B, C are three landmarks; B is 5 miles S. 40° E. from A, and C is 5 miles S. 36° W. from B. An observer on a ship finds that B is directly behind C, and that A bears N. 15° E. After sailing for half-an-hour directly towards A, the observer finds that C bears due E. Find the speed of the ship.

40. A rectangular table is 6 ft. by 4 ft. A point on the table is 37″ and 23″ from two adjacent corners. Find its distances from the other corners.

Problems in Three Dimensions.

41. The height of an inaccessible tower is required. Two stations P, Q are chosen, Q being 220 ft. due E. of P and in the same horizontal plane. The tower bears N. 49° E. from P and N. 57° W. from Q. Also the angle of elevation of the tower from P is 28°. Find the height of the tower.

42. A level road runs N. and S. along the top of a cliff. From two points A and B in the road, a boat on the sea bears S.E. and S. 61° E. respectively, and the angle of depression of the boat from B is 15°. If A is N. of B and AB = 100 yards, find the height of the cliff.

43. From two boats P, Q the bearings of a tower are respectively N. 20° E. and N. 30° W. The angles of elevation of the top of the tower are respectively 10° and 20°. Find the bearing of P from Q.

44. P, Q, R are in the same horizontal plane. P is the foot of a tower. The bearings of Q and R from P are respectively S. 40° E. and S. 32° W. From the top of the tower the angles of depression of Q and R are respectively 10° and 20°. Find the bearing of R from Q.

45. From a ship the elevation of a balloon was observed to be 24° 30′ and its bearing to be N. 5° W. At the same time from a ship 3000 yards E. of the first the balloon bore N. $45\frac{1}{2}$° W. What was the height of the balloon?

46. A, B, C are three points in a straight line in the horizontal plane passing through the foot of a tower PQ. A is 100 yards due S. of Q, C is 200 yards due E. of Q, B is S.E. of Q. Find the distance BQ. Also if the elevation of P from A is 18°, what is it from B and from C?

47. A square tower has pinnacles of equal height at the corners. From a point in the horizontal plane on which the tower stands, three of these are visible and their elevations are 35°, 45°, 35°. If a side of the base of the tower is 45 ft., what is the height of the top of a pinnacle above the horizontal plane?

48. Two observers A, B, 1475 yards apart on a horizontal plane, take simultaneously the altitude of a kite K as 15° 10′ and 13° 20′ respectively. At the same time the angles KAB and KBA are found to be 74° 41′ and 58° 19′. Find the height of the kite above the ground. Show that the height may be found in two ways and that the results agree within a yard.

49. From the top of a cliff 500 ft. high, a man observes two ships bearing respectively 21° and 47° E. of S. whose angles of depression are respectively 8° and 14°. Find the distance between the ships.

50. ABC is a plane triangle in which BC, CA, AB are 3, 4, 5 ft. respectively. At A, B, C perpendiculars AP, BQ, CR are drawn to the plane their lengths being 6, 7, 8 ft. respectively. Find the angles of the triangle PQR.

CHAPTER X

DEGREE OF APPROXIMATION

79. In using 4-figure tables to work out results from certain data, it is important to be able in any case to estimate by how much the calculated result may be in error. Possible errors in the final result are due to two causes:

(1) the numbers taken from the tables are known to be true only to a certain degree of accuracy;

(2) the data themselves are as a rule only approximate.

We shall not attempt to establish any general rules, but a few examples will show how any case may be dealt with on its own merits.

Ex. 1. Find the value of 1200 sin 40°.

The tables give $\sin 40^\circ = \cdot 6428$

and $1200 \times \cdot 6428 = 771 \cdot 36.$

Now all we know is that $\sin 40^\circ > \cdot 64275$ and $< \cdot 64285$.

Thus $1200 \sin 40^\circ > 771 \cdot 30$ and $< 771 \cdot 42$.

Ex. 2. Find an acute angle A so that

$$\sin A = \frac{39}{452} \sin 58^\circ.$$

We have

No.	log
39	$1 \cdot 5911$
sin 58°	$\bar{1} \cdot 9284$
	$1 \cdot 5195$
452	$2 \cdot 6551$
sin A	$\bar{2} \cdot 8644$

Since each of the logs we have taken out may be as much as ·00005 in error, all we can say is

$$\log \sin A = \bar{2}\cdot 8644 \pm \cdot 00015.$$

∴ to the nearest minute $A = 4^\circ\ 12'$, for the difference columns show that if A differed from $4^\circ\ 12'$ by as much as half a minute, log sin A would differ from $\bar{2}\cdot 8647$ by something like ·0008.

Ex. 3. In a triangle $b = 727$; $c = 864$; $A = 16^\circ$; find a.

$$a^2 = b^2 + c^2 - 2bc \cos A.$$

Proceeding as in § 74, Ex. 2, p. 112, we have

$$b^2 = 528500 \text{ (square table)},$$
$$c^2 = 746500 \text{ (square table)},$$
$$b^2 + c^2 = 1275000,$$
$$2bc \cos A = 1207000.$$
$$\therefore\ a^2 = 68000,$$
$$\therefore\ a = 260\cdot 8.$$

No.	log.
2	0·3010
727	2·8615
864	2·9365
cos 16°	$\bar{1}\cdot 9828$
$2bc \cos A$	6·0818

Now, each of the numbers taken from the square table is liable to an error ± 50.

Each of the logs is liable to an error ± ·00005,

so that 6·0818 .... ± ·0002,

i.e. log $2bc \cos A$ is between 6·0816 and 6·0820.

Thus $b^2 + c^2$ lies between 1274900 and 1275100,

and $2bc \cos A$ ....*1206000 and 1208500,

∴ $b^2 + c^2 - 2bc \cos A$ .... 66400 and 69100,

∴ a .... 257·65 and 262·95.

Ex. 4. The sides BC, AC and the angle B of a triangle are found by measurement to be 861 ft., 736 ft., and 58° respectively. If the sides are known to be correct only to the nearest foot and the angle to the nearest degree, find within what limits the angle A (supposed acute) may lie.

* It is safer when difference columns are used to allow for a possible error of ·0001.

We have
$$a = 861 \pm \cdot 5,$$
$$b = 736 \pm \cdot 5,$$
$$\mathrm{B} = 58^\circ \pm \cdot 5^\circ,$$
$$\sin \mathrm{A} = \frac{a \sin \mathrm{B}}{b}.$$

In working out $\log \sin \mathrm{A}$ we shall get the least possible value by making $\log a$, $\log \sin \mathrm{B}$ as small and $\log b$ as large as possible, and similarly we shall get the greatest possible value by making $\log a$, $\log \sin \mathrm{B}$ as large and $\log b$ as small as possible.

No.	log	No.	log
860·5	$2 \cdot 9348 \pm \cdot 0001$	861·5	$2 \cdot 9353 \pm \cdot 0001$
sin 57° 30′	$\bar{1} \cdot 9260 \pm \cdot 00005$	sin 58° 30′	$\bar{1} \cdot 9308 \pm \cdot 00005$
	$2 \cdot 8608 \pm \cdot 00015$		$2 \cdot 8661 \pm \cdot 00015$
736·5	$2 \cdot 8672 \pm \cdot 0001$	735·5	$2 \cdot 8666 \pm \cdot 0001$
	$\bar{1} \cdot 9936 \pm \cdot 00025$		$\bar{1} \cdot 9995 \pm \cdot 00025$

i.e. $\log \sin \mathrm{A} > \bar{1} \cdot 99335$ and $< \bar{1} \cdot 99975$,

i.e. A is between something under 80° and something above 88°.

EXERCISES XLIII.

(*a*) Considering only errors due to tables.

Find in each case between what limits the quantity to be calculated may lie, assuming the data to be correct.

1. p. 13, Ex. 30.

2. p. 45, Ex. 48.

3. p. 45, Ex. 56.

4. p. 80, Exs. XXVII. 4.

5. p. 54, Ex. 2. Find between what limits the number in the last column may lie in each of the last three lines.

6. The same for p. 56, Ex. 8.

7. Account for the discrepancy between the results of § 72, Ex. 3 and those of § 76, Ex. 3.

8. Within what limits may b lie in § 74, Ex. 2?

9. Within what limits may A lie in § 75, Ex. 2.

(*b*) Considering both sources of error.

Find in each case between what limits the quantity to be calculated may lie, the data being liable to error as stated.

10. p. 7, Ex. 1 (iii) and (iv). The distance is correct to the nearest foot and the angle of elevation to the nearest degree.

11. p. 7, Ex. 2 (iv). There is a possible error of 1 foot in the distance and of 0·1° in the angle.

12. p. 13, Ex. 31. The height of the tree is correct to the nearest foot and each angle to the nearest degree.

13. p. 15, Ex. 15 (i). The height of the cliff is known within 5 feet and the distance of the boat within 50 yards.

14. p. 22, Ex. 14. The slope is known to the nearest degree.

15. p. 45, Ex. 53. The angles are not more than 0·1° in error and the distance between the buoys is correct to the nearest yard.

16. Two sides of a right-angled triangle are 97 feet and 94 feet, each correct to the nearest foot. Find the limits between which the third side may lie if (i) the given sides contain the right angle, (ii) one of the given sides is the hypotenuse.

17. pp. 120, 121, worked Ex. 1. The length of AB is correct to the nearest foot and the angles of elevation are not more than 0·1° in error.

18. p. 12, Ex. 24. The distance of B from A may be a tenth of a mile in error and each of the bearings is known to the nearest degree.

19. p. 126, Ex. 15. The distance of Newhaven from Portsmouth is correct to the nearest mile and each bearing to the nearest degree.

20. p. 86, Ex. 9. The distance may be a mile and the course half a degree in error. Find the distances N. and W. taking

Distance	86	86	86	87	87	87	88	88	88
Courses	$33\frac{1}{2}°$	34°	$34\frac{1}{2}°$	$33\frac{1}{2}°$	34°	$34\frac{1}{2}°$	$33\frac{1}{2}°$	34°	$34\frac{1}{2}°$

Plot the corresponding positions on squared paper, using as large a scale as is convenient. You will see that all that is known about the position of the ship is that it is somewhere inside a certain quadrilateral.

MISCELLANEOUS EXERCISES.

A.

1. Construct the acute angle whose sine is ·7 and from the figure find its cosine and tangent.

2. Use tables to find the value of

$$\sin A + \sin 2A - \sin 3A \text{ when } A = 18^\circ.$$

3. ABC is a triangle right-angled at A. AD is perpendicular to BC and DE to AC.

Prove $\frac{DE}{AB} = \sin^2 B$ and $\frac{EC}{BC} = \sin^3 B$.

4. The radius of a circle is 10 cms. Find the lengths of the arcs cut off by a chord 7 cms. long.

[If AB is an arc of a circle, centre O, the arc AB is the same fraction of the whole circumference that the angle AOB is of four right angles.]

5. B is 3200 yards N. 70° E. from A; P is 1700 yards N. of A; Q is 3000 yards N. of B. Find the distance and bearing of Q from P.

B.

1. Solve the triangle in which $A = 90^\circ$, $B = 57^\circ$, $a = 12$.

2. Draw the graph of $\sin A$ from $A = 45^\circ$ to $A = 46^\circ$ on a large scale using the values given for sin 45°, sin 45·1°, sin 45·2°, etc. From the graph find sin 45° 38′.

3. Given $\cos A = \frac{1}{3}$, calculate $\sin A$ and verify your result from the tables.

4. Find the vertical angle of a cone, the diameter of the base being 9″ and the slant height 12″.

5. A rectangular vertical target is fixed in the ground facing due S. It is 12 ft. high and 8 ft. wide. Find the area of its shadow when the sun's altitude is 40°, if the sun is 15° from the S.

C.

1. Construct the angle whose tangent is 3·3 and find its sine from your figure.

2. Solve the triangle in which $C=90°$, $a=5$, $c=3b$.

3. AB is a chord of a circle, OC the radius which bisects AB, x is the length of the chord AB and y that of the chord AC. An approximate formula for the length of the arc ACB is $\frac{8y-x}{3}$.

Verify this formula

(i) when the radius is 5 cms., $\angle AOB=40°$,

(ii) when the radius is 2 inches, $\angle AOB=70°$.

4. In a quadrilateral ABCD, $A=125°$, $B=130°$, $DA=4''$, $AB=5''$, $BC=6''$. Find the length of the projection of DC on AB.

5. A wire is stretched tightly between two points in a horizontal line 20 ft. apart. A weight is hung from the middle point and each portion of the wire now makes an angle of 5° with the horizontal. How much has the wire been stretched, and how far is the middle point below its original position?

D.

1. On a large scale draw the graph of cos A between $A=27°$ and $A=28°$, using values given for cos 27°, cos 27·1°, etc. From the graph write down the angle whose cosine is ·8865.

2. Solve the triangle in which $a=b$, $c=16$, $A=37°$.

3. Given $\tan A=\frac{15}{8}$, calculate $\cos A$.

4. Two opposite angles of a kite are 30° and 130°. The diagonal joining the equal angles is 8 cms. Find the second diagonal.

5. A road runs from a point A on the 200 ft. contour line to a point B on the 300 ft. contour line. The slope of the road is 1 in 16. How long is it?

E.

1. Verify the formula $\sin 2A=2\sin A\cos A$ when

(i) $A=30°$, (ii) $A=45°$, (iii) $A=23°$.

2. The acute angle of a rhombus is 41° and one diagonal is 2″ longer than the other. Find the diagonals.

3. Show that the sine of any acute angle is less than the tangent.

4. A is any point in the line XY. B is a point outside XY so that $AB = 4''$ and $\angle BAY = 35°$. Find the radius of a circle which passes through B and touches XY at A.

5. ABCDE is a regular pentagon. It is folded so that A comes to the mid-point of CD. How far is B from its original position if the side of the pentagon is 10 cms.?

F.

1. Construct an angle A, such that $\sin A = 2 . \cos A$. Look out the sine and cosine of the angle you obtain and verify your result.

2. Solve the quadrilateral ABCD, given

$$AB = 2'',\ BC = 3'',\ CD = 4'',\ B = 90°,\ C = 90°.$$

3. Draw the graph of $\sin A + \cos A$ between $A = 0°$ and $A = 90°$. For what values of A does $\sin A + \cos A = 1$? What is the greatest value of $\sin A + \cos A$ between these limits?

4. In a circle of radius 5 cms. a triangle is inscribed whose angles are 47°, 64°, 69°. Find the perimeter of the triangle.

5. From two points 3 kilometres apart on a straight road running N. and S. a building bears S. 37° W. and N. 43° W. Find the distance of the building from the road.

G.

1. Solve the triangle in which

$$A = 90°,\ b = 13{\cdot}28,\ c = 19{\cdot}26.$$

2. Find the value of an acute angle A such that

$$\sin A = \frac{\sqrt{(a+b+c)(b+c-a)(c+a-b)(a+b-c)}}{2bc},$$

when $a = 3$, $b = 4$, $c = 5$.

3. Find the value of $\sin^3\theta . \cos^2\theta$ when $\theta = 48°$.

4. In a circle of radius 10 cms., chords AB, BC, CD are placed successively, their lengths being 7, 9, 11 cms. Find the length of AD.

5. In a map drawn on a scale of an inch to a mile, the points in which a path crosses the contour lines of 1500 and 1750 ft. are $\cdot 2''$ apart. Find the average angle of slope of the path.

H.

1. In a triangle ABC, $C = 90°$, $a = 5$, $b = 12$. Find the values of

$$\frac{\sin A}{1 + \cos A} \text{ and } \frac{\sin A}{1 - \cos A}.$$

2. Draw the graph of $\sin A - \cos A$, from $A = 0°$ to $A = 90°$.
For what value of A does $\sin A - \cos A = \cdot 3$?

3. Find the magnitude and direction of the resultant of two equal forces each of 17·9 lbs. wt. acting at an angle of 76°; also of two forces each of P lbs. wt. acting at an angle a.

4. AC is a diameter of a circle. On the tangent at A, AB is taken equal to $3 \cdot AC$. OD is a radius making an angle of 30° with OC. DP is perpendicular to OC. Show that approximately PB = circumference of the circle.

5. Two pillars of equal height stand on either side of a road 100 ft. wide. At a point in the road in a line between the pillars their angles of elevation are 35° and 40°. Find the height of the pillars and the position of the point of observation.

I.

1. ABC is a triangle in which $A = 90°$, $AB = AC = 1''$. AB is produced to D so that $BD = BC$. Without using any trigonometrical table calculate the lengths of AD and CD. Hence calculate the sine, cosine and tangent of $22\frac{1}{2}°$ and $67\frac{1}{2}°$.

2. Verify that $\sin A \cos B + \cos A \sin B = \sin(A + B)$
(i) when $A = 36° 52'$, $B = 53° 8'$; (ii) when $A = 25°$, $B = 40°$.

3. Prove from the definitions $\tan A + \cot A = \sec A \operatorname{cosec} A$.

4. Two circles, radii 3 and 5 cms., have their centres 11 cms. apart. Find the angles made with the line of centres by the common tangents.

5. From a ship steaming due N. at 12 knots, the bearing of a lighthouse was N. 30° W. 5 minutes later it was N. 60° W. Find the distance of the ship from the lighthouse at each observation.

J.

1. The number of nautical miles in the distance between two points on the earth's surface which have the same latitude but differ by 1° in longitude is $60 \times$ cosine of latitude. Find (i) the distance between two points A and B, in the same latitude, 53° N., the longitude of A being 10° W. and of B 35° W., (ii) in what latitude a ship would change her longitude 5° by sailing 130 miles along a parallel.

2. Draw the graph of $2 \sin\phi + 3 \cos\phi$ between $\phi = 0°$ and $\phi = 90°$ and find for what value of ϕ the expression has its greatest value between these limits. Also find what value of ϕ makes the given expression equal to 2·5.

3. Solve the triangle in which $a=b=39{\cdot}21$, $c=20{\cdot}28$.

4. A regular pentagon and a regular hexagon have the same area. If the side of the pentagon be 2″, what is the side of the hexagon?

5. From the top of a hill 529 ft. above the level of a lake the angle of elevation of the top of a mountain on the other side is 11° 30′, and the angle of depression of the image in the lake is 16° 30′. Find the height of the mountain.

K.

1. Given
$$\sin\frac{\text{A}}{2}=\sqrt{\frac{(s-b)(s-c)}{bc}},$$
where
$$s=\frac{a+b+c}{2},$$
find A when $a=24$, $b=24$, $c=32$. Also with the same values for a and c find B if $\cos \text{B}=\frac{c}{2a}$.

2. ABCD is a square of side 5 cms. Through A a line is drawn outside the square making an angle of 28° with AB. Find the length of the projection of each side and diagonal of the square on this line.

3. ABC is a triangle in which BC=20 cms., B=38°, C=52°. AD is perpendicular to BC. Find the lengths of AD, BD, CD.

4. In a triangle ABC, BC=4″, B=43°, C=66°. Find the radius of the circle circumscribing the triangle.

5. A weight is suspended by means of two equal strings fastened to two points A, B in the same horizontal plane. Each string is inclined at an angle of 10° to AB. If the weight is increased so as to increase this angle to 20°, each string is stretched 2″. Find the distance AB.

L.

1. Draw a semi-circle whose diameter AB is 10 cms. At A, B draw tangents AM, BN. Draw AP making ∠BAP=32° meeting the circumference in P and BN in C. Produce BP to meet AM in D. Measure AP, BP, AC, BC, AD, BD; thence write down sin 32°, cos 32°, tan 32°, cot 32°, sec 32°, cosec 32°. Compare with the tables.

2. ABC is a triangle in which A=90°, B=30°, AC=1″. AB is produced to D so that BD=BC. Calculate the lengths of AD, CD and hence calculate sine, cosine and tangent of 15° and 75°.

3. In a triangle ABC, $A=90°$, $BC=16$ cms., AD (the perpendicular from A to BC) $=6$ cms. Solve the triangle.

4. P is a point outside a circle centre O. PQ is perpendicular to OP and 5 cms. in length. The tangent from P to the circle makes an angle of 42° with OP: that from Q makes 38° with OQ. Find the radius of the circle.

5. A tower PQ is S. of A and S. 63° W. from B. The angle of elevation of P from A is 35°. If B is 325 ft. due E. of A, find the height of the tower.

M.

1. From the tables find the values of $\sin A + 3\cos A$ when $A=35°$, 36°, 37°, 38°.

Plot your values on squared paper, taking 2″ horizontally to represent 1°, and 1″ vertically to represent ·01. Join consecutive points by straight lines. From your graph find to the nearest minute for what value of A, $\sin A + 3\cos A = 3$.

2. Find an acute angle θ so that $\tan\theta = \sqrt{2\sin 40° \,.\, \cos 20°}$.

3. The angle between the tangents from a point to a circle is 40°. The angle between the tangents from a point 2″ nearer to the centre is 100°. Find the radius of the circle.

4. The perimeter of a square is twice that of a regular pentagon. Find the ratio of their areas.

5. A tower is W. of A and N. of B. A is N. 35° E. from B. If the angle of elevation of the tower from A is 40°, what is its angle of elevation from B?

N.

1. Given $\tan\theta = \dfrac{b-c}{b+c}\cot\dfrac{A}{2}$, find θ if $b=17{\cdot}29$, $c=8{\cdot}46$, $A=63°\,48'$.

2. Find the value of $\sqrt[3]{\sin 162° \times \tan^2 58°}$.

3. Draw the graph of $2\sin\dfrac{A}{2} + \sin A$ from $A=0°$ to $A=180°$, and find for what value of A the expression has its greatest value between these limits.

4. ABC is an equilateral triangle whose side is 6″. D is a point in BC so that $BD=\dfrac{1}{3}BC$. Find the length of AD.

5. A ship is sailing at constant speed on a fixed course. An observer at O finds that the ship is due E. of him at 1 o'clock, S. 30° E. at 1.30 and due S. at 1.50. If the successive positions of the ship are P, Q, R, express (i) $\frac{OQ}{PQ}$ in terms of angle P; (ii) $\frac{OQ}{QR}$ in terms of angle R; (iii) $\frac{PQ}{QR}$ in terms of angle R. Hence find the ship's course.

O.

1. Find the value of $\cos 20° \times \cos 40° \times \cos 80°$.

2. ABC is a triangle in which $A = 90°$, $AB = 4''$, $AC = 3''$. On BC and CA squares BCDE and CAFG are described externally. Find the length of GD.

3. The sides of a rectangle are 39·4 and 53·7 cms. Find the acute angle between the diagonals.

4. The road from A to D consists of three parts AB, BC, CD of equal length. The inclination to the horizon of AB is 2° 30′, of BC 4°, of CD 3° 15′. If D is 150 ft. above A, find the distance from A to D.

5. What does the formula $a^2 = b^2 + c^2 - 2bc \cos A$ become (i) when $A = 0°$, (ii) when $A = 180°$?

Draw figures to explain the results.

P.

1. A ship steams E. along a parallel of latitude starting from long. 10° E., lat. 48° N. What will be her position after 12 hours, if she is doing 15 knots? [See J. 1.]

2. APB is a semi-circle, centre O, diameter AB, P any point on the circumference, PN perpendicular to AB. AP, BP are joined. Calling the angle PAB, α, write down $\sin \alpha$ from $\triangle$ PNA, and $\cos \alpha$ from $\triangle$ PAB. Hence prove $2 \sin \alpha \cos \alpha = \frac{PN}{OP}$. Deduce the formula $\sin 2\alpha = 2 \sin \alpha \cos \alpha$.

3. If the side of a square inscribed in a certain circle be 2″, find the side of a regular octagon inscribed in the same circle.

4. Find the value of $\frac{\sin(\alpha + \beta)}{\cos \alpha \cos \beta}$ when $\alpha = 28°$, $\beta = 78°$.

5. A and B are two stations 123 yards apart, A being S. 33° W. from B. From A the bearings of C and D are N. 10° E. and S. 75° E. respectively. From B, C bears N. 51° W. and D bears S. 10° E. Find the distance and bearing of D from C.

Q.

1. Find the value of $(b+c)\cos\theta$, if $\sin\theta = \frac{\sqrt{bc}}{b+c}$, where $b=29\cdot3$, $c=18\cdot7$.

2. Find the area of a triangle ABC, given that $B=47°$, $C=78°$, and the perpendicular from A to $BC=12$ cms.

3. A piece of cardboard in the form of a regular pentagon ABCDE of side 8″ stands against a vertical wall with the side AB on the ground. It is turned about A until AE is on the ground. Find the distance between the first and the second position (i) of B, (ii) of C.

4. Find the angle between the lines joining (i) (2, 3) and (4, 4), (ii) (2, 2) and (4, 5). [The same scale is used along both axes.]

5. A circular tower is surrounded by a moat 20 ft. wide. A man on the outer rim of the moat notices that two trees which are just visible past the tower subtend at his eye an angle of 42°. Find the diameter of the tower.

R.

1. Draw the graph of log sin A from $A=0°$ to $A=90°$.

2. The time the sun takes to rise or set at an equinox in latitude l is $\left(\frac{1}{15}\,\mathrm{D}\sec l\right)$ seconds, where D″ is the sun's angular diameter. Find the time taken by the sun to rise in London, given that the sun's angular diameter is 32′, and the latitude of London 51° 32′ N.

3. Find the area of a quadrilateral ABCD in which

$$AB=2'',\ BC=2\cdot5'',\ CD=3\cdot2'',\ B=110°,\ C=124°.$$

4. APB is a semi-circle, centre O, diameter AB. P is any point on the circumference, and PN is perpendicular to AB. Calling the angle PAB, α, find $\sin\alpha$ (i) from $\triangle$PNB, (ii) from $\triangle$PAB; also find $\cos\alpha$ (i) from $\triangle$PNA, (ii) from $\triangle$PAB.

Hence prove $\sin^2\alpha = \frac{NB}{AB}$, and $\cos^2\alpha = \frac{AN}{AB}$. Deduce $\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$.

5. A sphere subtends at a man's eye which is on the same level as the centre of the sphere an angle of 34°. He walks 5 ft. nearer to the sphere and the angle is now 48°. Find the diameter of the sphere.

S.

1. The beam of an ordinary balance is a foot long. If it swings through an angle of 20°, how far will each scale-pan be from its original position?

2. The angles of a triangle are 40°, 60°, 80°, and its area is 20 sq. ins. Find the sides.

3. CP is a crank 5″ long, rotating about C; PQ is a rod 16″ long, whose end Q moves in a straight line, the perpendicular CM to which from C is 7″. Find (i) the distance QM when $\angle PQM = 30°$, (ii) the greatest angle between PQ and QM.

4. A sheet of paper is in the shape of a rectangle ABCD in which AB = 8″, BC = 10″. It is folded so that B comes on AD and the line of the crease passes through C. Find where the line of the crease meets AB.

5. A ladder inclined at an angle of 42° to the horizontal just reaches the top of a wall. The foot is moved 10 ft. nearer to the wall, and the top then projects 7 ft. beyond the wall. Find the length of the ladder and the height of the wall.

T.

1. In each case find an acute angle θ so that

(i) $\sin\theta = \sin 30° + \sin 10°$;

(ii) $\sin\theta = 2\sin 15°$;

(iii) $\sin\theta = \frac{1}{2}\sin 30°$.

2. A pyramid, vertex V, has a square base ABCD whose side is 10 cms. If the angle AVB is 30°, find the height of the pyramid.

3. ABC is an equilateral triangle. A straight line XAY is drawn through A outside the triangle so that the angle BAY = 40°. What is the angle CAX? What is the acute angle between CB and XY? By projecting the sides of the triangle on XY, prove that $\cos 40° + \cos 80° = \cos 20°$.

Verify this result from the tables.

4. The sides of a triangle are 28·3, 39·4, 43·8 cms. Find the three altitudes.

5. Looking out of a window, with his eye 15 ft. above the road, an observer finds that the angle of elevation of the top of a telegraph post is 17° 19′, and the angle of depression of the bottom of the post is 8° 32′. Find the height of the post and its distance from the observer.

U.

1. If $\tan\theta = \dfrac{u\sin\alpha - gt}{u\cos\alpha}$, find θ when $u=250$, $g=32$, $t=3$, $\alpha=35°$.

2. The sides of a triangle ABC are 36·8, 49·2, 53·1 mms. A point O is equidistant from A, B, C. Find the angles subtended at O by the sides of the triangle.

3. A chord AB of a circle, whose diameter is d, subtends an angle 2θ at the centre. Find the heights of the arcs into which AB divides the circumference.

4. The radii of the ends of a frustum of a right circular cone are 12 and 8 ft., and the thickness is 15 ft. Find the vertical angle of the cone of which it is a frustum.

5. B is 4 miles N. 50° E. from A. From A a length AC of 2000 yards is measured in a N.W. direction, and the bearings of a point P from A and C are found to be N. and N. 25° E. respectively. Find the distance and bearing of P from B.

V.

1. A semi-circle whose diameter AB is $2a$ inches is drawn on a vertical blackboard with AB horizontal. PQ is a chord parallel to AB and subtends an angle 2θ at the centre. On PQ as diameter a semi-circle is drawn on the side remote from AB. If R is the highest point of this semi-circle, find its distance from AB. Where must PQ be drawn so that R shall be at the greatest possible distance from AB? [See F. 3.]

2. ABCDE is a regular pentagon. If AB=34″, show that AC=55″ nearly.

3. Resolve a force of 314 gms. wt. into two components whose directions make angles of 35° and 75° with the direction of the given force.

Also resolve a force of R lbs. wt. into two components whose directions make angles α, β with the direction of the given force.

4. In a circle of radius 10 cms., a chord AB is 18 cms. long. Calculate the lengths of the parts into which AB is divided by a diameter which cuts it at an angle of 40°.

5. Two persons P, Q, stationed on a coast which runs E. and W., observe a ship when it is due N. of P, and again when it is due N. of Q. In the former case it is N.W. from Q, and in the latter N. 30° E. from P. Find the direction in which the ship is sailing.

W.

1. If
$$\frac{1}{r^2}=\frac{\cos^2\theta}{a^2}+\frac{\sin^2\theta}{b^2},$$
find r when $a=4{\cdot}63$, $b=2{\cdot}41$, $\theta=35°$.

2. Find graphically the greatest and least values of $4\cos x+3\sin x$, for values of x between 0 and 180°, and the corresponding values of x.

3. Two chords of a circle whose lengths are a and b subtend angles A and B at the circumference. Prove $\frac{a}{\sin A}=\frac{b}{\sin B}$.

4. Two circular discs of radii 10 cms. and 7 cms. lie on a table with their centres fixed 25 cms. apart. Find the length of a cord which will pass round both discs, (i) if the strings cross one another forming a figure of eight, (ii) if they do not cross.

5. A church spire is seen in a direction due N. of a station A at an elevation of 17°, and from a station B 120 ft. due E. of A, the spire bears N. 23° W. The two stations and the foot of the spire being at the same level, find the height of the spire.

ANSWERS

EXERCISES III. Page 7.

1. (i) 57·74 ft., (ii) 86·60 ft., (iii) 72·8 ft., (iv) 219·8 ft.
2 (i) 119·18 ft., (ii) 51·96 ft., (iii) 320 ft., (iv) 55·96 ft.
3. (i) 824·2 ft., (ii) 536·1 ft., (iii) 571·2 ft. **4.** 169·5 ft.
5. 112·0 ft.

EXERCISES IV. Page 9.

1. (i) 70·02, (ii) 73, (iii) 643·35, (iv) 38160, (v) 161·20, (vi) 36·68.
2. (i) 142·81, (ii) 73, (iii) 1119·6, (iv) 188·18, (v) 8·332, (vi) 93·25.
3. 5·712 cms. **4.** 22·21 cms. **5.** 1·7321″. **6.** 2·3096″.
7. 21·98 cms. **8.** 12·867 cms. **9.** 5·774″.
10. 4·142 cms., 5·858 cms. **11.** 55°, 55°, 4·284″. **12.** 16·782″.
13. 11·425″. **14.** 14·165″. **15.** 23·835″.
20. 17·321, 3·640, 8·391. **24.** 11·72 m. **25.** 7·002 m.
26. 114·45 yds., N. 41° 09′ E. **27.** 39 25 ft. **28.** 25·0 ft.
29. 311·1 ft. **30.** 24·40 ft. **31.** 178·63 ft.

EXERCISES V. Page 14.

1. (i) 35°, 55°, (ii) 41° 01′, 49°59′, (iii) 73°, 17°, (iv) 29°45′, 60°15′, (v) 36°52′, 53°08′, (vi) 82° 21′, 7°39′.
8. 73°18′, 73°18′, 33°24′. **9.** 35°. **10.** 77°19′, 102°41′.
11. N. 63°26′ E. **12.** N. 54°28 W.
13. (i) 45°, (ii) 153°26′, (iii) 135°, (iv) 60°15′, (v) 11°19′.
14. (i) 45°, (ii) 63°26′, (iii) 36°52′.
15. (i) 3°35′, (ii) 9°28′, (iii) 6°51′. **16.** 47°.

EXERCISES VI. Page 19.

2. $\left.\begin{array}{l}\text{Sin } 35° \text{ between } 0·569 \text{ and } 0·578\\ \text{Cos } 35° \text{ between } 0·814 \text{ and } 0·825\end{array}\right\}$.

EXERCISES VIII. Page 21.

1. (i) 5·736, 8·192, (ii) 5·56, 5·75, (iii) 96·13, 27·56, (iv) 61·80, 32·86, (v) 0·782, 4·938, (vi) 9·46, 7·39.
3. ·97, ·87, ·68. **4.** 55°, 55°, 9·18″. **5.** 28·19 ft., 10·26 ft.

6. 2·21 ft., 4·74 ft. **7.** 5·25 m., 2·91 m. **8.** 8·192 m., 5·736 m.
9. 4·93 m., 6·30 m. **10.** 3·72 m. **11.** 11·47 cms., 16·38 cms.
12. 13·11 cms. **13.** 4·510″. **14.** 87·2 yds.
15. 5·18 m., 19·32 m.
16. (i) 24, (ii) 15, (iii) 18, (iv) 10, (v) 16, (vi) 15, (vii) 16, (viii) 48·3, (ix) 11·93, (x) 14·72.

EXERCISES IX. Page 24.

10. 0·75, 48°36′. **11.** 0·4, 66°25′.
12. (i) 36° 52′, 53° 08′. (ii) 44° 25′, 45° 35′, (iii) 58°, 32°, (iv) 60°, 30°, (v) 53° 08′, 36° 52′, (vi) 76° 21′, 13° 39′.
13. 34° 55′, 72° 32½′, 72° 32½′. **14.** 45° 35′. **15.** N. 36° 52′ E.
16. 2° 52′. **17.** 33° 55′. **18.** 47° 10′. **19.** 38° 56′.
20. 18° 12′.

EXERCISES X. Page 26.

1. 5·299 m., 8·480 m. **2.** 2·5″ on AB, 3·830″ on AC.
3. Same as Ex. 2. **4.** 5·80″. **5.** 0·50″.
6. 8·064 m., 5·571 m. **7.** 990·3 ft. **8.** 51° 19′.
9. 7″, B = 36° 52′, C = 45°, A = 98° 08′. **10.** 9·434″, N. 58° E.
11. 3·464, 3·214, 6·678; 2, 3·830, 5·830. **12.** 8·87 m., N. 41·1° E.
13. 6·92 m., N. 15·3° W. **14.** 5·73 lbs. wt., 4·01 lbs. wt.
15. 12·81 lbs. wt., N. 32° 20′ E.

EXERCISES XI. Page 27.

1. 25; $\frac{7}{25}, \frac{24}{25}, \frac{7}{24}, \frac{24}{25}, \frac{7}{25}, \frac{24}{7}$.
3. 25, 52, 60, 39 cms.; 104° 15′, 59° 29′, 75° 45′, 120° 31′.
11. $\frac{15}{17}, \frac{8}{15}$. **14.** (i) $\frac{3}{5}, \frac{4}{5}$, (ii) $\frac{12}{13}, \frac{12}{5}$, (iii) $\frac{9}{41}, \frac{40}{9}$.
24. 65° 42′.

EXERCISES XII. Page 39.

1. (ii) A = 28° 04′, B = 61° 56′, c = 3·4; (iii) B = 36°, b = 2·91, c = 4·94; (iv) B = 47°, a = 3·41, b = 3·66.
2. A = 36° 52′, B = 53° 08′, c = 5. Area = 6.
3. B = 67° 23′, C = 22° 37′, b = 12. Area = 30.
4. C = 50°, a = 3·214, c = 3·830. Area = 6·155.
5. A = 25° 48′, a = 4·834, c = 11·107. Area = 24·17.
6. B = 10° 23′, C = 79° 37′, c = 60. Area = 330.
7. A = 12° 41′, C = 77° 19′, b = 41. Area = 180.
8. C = 74° 45′, b = 3·16, c = 11·58. Area = 18·27.

9. $A=28°04'$, $C=61°56'$, $b=17$. Area $=60$.
10. $B=18°51'$, $a=23·43$, $c=24·76$. Area $=93·74$.
11. $A=14°15'$, $B=75°45'$, $b=63$. Area $=504$.
12. $C=62°$, $b=3·29$, $c=6·18$. Area $=10·16$.
13. $A=36°42'$, $c=5·37$, $b=6·69$. Area $=10·73$.

EXERCISES XIII. Page 39.

1. 320·1 ft., 72° 39′, 577 ft. 2. 113·5 yds., 72·3 yds.
3. 10 ft., 53° 08′. 4. 85° 14′, 69° 27′, 23° 12′.
5. 22·33 ft., 6·38 ft., 68·06 ft. 6. 1° 09′. 7. 2·736 ft., 2·571 ft.
8. 6·21 ins., 7° 26′. 9. 36° 52′. 10. 92·60 ft.
11. 943 ft., 2486 ft. 12. 128 ft., 566 ft. 13. 23·94 ft.
14. 22·36 m., N. 43° 26′ W. 15. 15·62 m., N. 50° 12′ W.
16. 2° 27′. 18. 9·92 ft. 19. 8·04 ft. 20. 48·38 ft.
21. 81° 47′. 22. 3·354 ft., 73° 24′. 23. 23° 35′.
24. 2·828″, 19° 28′. 25. 1·786″.
26. 8·192 cms., 5·736 cms., 9·397 cms. 27. 18° 26′. 28. 1·618″.
30. 10 ft., 36° 52′. 31. 41° 24′, 97° 12′.
32 3·306 lbs. wt., 56° 19′, 33° 41′. 33. 18·79 lbs. wt., 3·47 lbs. wt.
34. 102° 38′, 128° 41′, 128° 41′. 35. N. 78° 41′ E.
36. N. 13° 27′ W. 37. 11·14 cms., 51° 03′.
38. $AC=10·83$ cms., $BC=11·50$ cms. 39. 3° 20′, 226 yds.
40. 4° 46′. 41. 2972 yds., 2676 yds. 42. 1928 yds., 2298 yds.
43. 43·30 yds. 44. 11·416″, $A=73°44'$, $C=44°02'$, $B=D=121°07'$.
45. 17·1 ft., 186·6 ft. 46. 6·86 ft. 47. 109·6 ft. 48. 686 ft.
49. 311·7 yds. 50. 53·13 ft. 51. 35·9 sq. ins.
52. $BP=100$ ft., 34·20 ft., 93·97 ft. 53. 102·6 ft.
54. ·8660 m. $=1524$ yds. 55. 46·99 sq. ins. 56. 73·0 ft.
57. 86·60 ft., 150 ft., 50 ft. 58. 4·62 m., 24½ mins. past 1.
59. 34·21 yds. 60. 23·49 sq. ins., 28·81 sq. ins. 61. 353·6 ft.
62. 113·4 ft. 63. 34° 32′. 65. 381 ft., S. 36° 07′ W.
66. 19° 06′. 67. 1245 ft. 68. 30 ft., N. 20° E.
69. 39·0 ft., N. 15° 16′ E. 70. 1664 ft.

EXERCISES XIV. Page 50.

4. (i) $b=6·40$, $c=8·12$, $B=52°$. (ii) $b=18·25$, $b=16·40$, $A=26°$.
(iii) $a=10·15$, $c=1·76$, $C=10°$. (iv) $a=19·07$, $b=14·82$, $C=39°$.
(v) $b=2·735$, $c=1·865$, $C=43°$. (vi) $b=61·55$, $c=64·72$, $A=18°$.
5. (i) $b=c=287·9$, $A=20°$. (ii) $a=c=8·51$, $B=72°$.
(iii) $a=b=8·28$, $C=150°$. (iv) $c=12·075$, $B=41°$, $C=98°$.
(v) $A=B=66°$, $a=b=24·59$. (vi) $B=40°$, $C=70°$, $a=c=14·62$.
(vii) $c=8$, $B=C=51°19'$, $A=77°22'$. (viii) $B=C=63°30'$, $a=5·354$.

6. 32·49 m. **7.** 3·53 m., 20·31 m. **8.** 13·95 m.
9. 2748 m., 2924 m. **10.** 178·5 ft., 217·9 ft.
11. 9·397, 18·79; 19·70. **12.** 7·182, 5·321, 7·331.
13. 7·607, 9·539, 15·097. **14.** 0·7645″. **15.** 2·1056 .
16. 10·515, 6·498. **17.** 7·4725 inches. **18.** 2·1284″.
19. (i) 3·090″, (ii) 5·321. **20.** 5·210″, 5·580″. **21.** 63° 31′.
22. 6·57 cms. **23.** 37·74 ft. **24.** 287·9 ft. **25.** 575·9 ft.
26. 567·13 ft.
29. 30·108 cms., 12·208 cms., 14·945 cms., 210·5 sq. cms.
36. $\frac{4}{5}, \frac{3}{4}, \frac{4}{3}, \frac{5}{4}, \frac{5}{3}$. **37.** $\frac{5}{13}, \frac{12}{13}, \frac{12}{5}, \frac{13}{12}, \frac{13}{5}$.

EXERCISES XV. Page 54.

1. AOB = 72°, AB = 5·878, Per. = 29·39.

2.

No. of sides	Angle at centre	Side	Perimeter	$\frac{\text{Perimeter}}{\text{Diameter}}$
3	120°	8·660	25·98	2·598
4	90°	7·071	28·28	2·828
5	72°	5·878	29·39	2·939
6	60°	5·000	30·00	3·000
8	45°	3·827	30·62	3·062
10	36°	3·090	30·90	3·090
20	18°	1·564	31·28	3·128
40	9°	0·785	31·40	3·140
100	3·6°	0·314	31·4	3·14

3. AOB = 72°, AB = 7·265, Per. = 36·325.

4.

Side	Perimeter	$\frac{\text{Perimeter}}{\text{Diameter}}$
17·321	51·96	5·196
10·000	40·00	4·000
7·265	36·33	3·633
5·774	34·64	3·464
4·142	33·14	3·314
3·249	32·49	3·249
1·584	31·68	3·168
0·787	31·48	3·148
0·314	31·4	3·14

7. AQ = 9·511, △AOB = 47·555, Area = 237·8.

8.

No. of sides	p	$\triangle$ AOB	Area	$\frac{\text{Area}}{(\text{radius})^2}$
3	8·660	43·30	129·9	1·299
4	10·000	50·00	200·0	2·000
5	9·511	47·56	237·8	2·378
6	8·660	43·30	259·8	2·598
8	7·071	35·36	282·8	2·828
10	5·878	29·39	293·9	2·939
20	3·090	15·45	309·0	3·090
40	1·564	7·82	312·8	3·128
100	0·628	3·14	314	3·14

9. AB = 14·53, $\triangle$ AOB = 72·65, Area = 36·325.

10.

No. of sides	$\triangle$ AOB	Area	$\frac{\text{Area}}{(\text{radius})^2}$
3	173·21	519·6	5·196
4	100·00	400·0	4·000
5	72·65	363·3	3·633
6	57·74	346·4	3·464
8	41·42	331·4	3·314
10	32·49	324·9	3·249
20	15·84	316·8	3·168
40	7·87	314·8	3·148
100	3·14	314	3·14

12. 5·774, 7·071, 8·506, 10, 13·065, 16·181, 31·96, 63·75, 159·2.
13. 2·887, 5, 6·882, 8·660, 12·071, 15·389, 31·57, 63·55, 159·1.
14. 43·31, 100, 172·05, 259·8, 482·84, 769·4, 3157, 12710, 79550.
15. 2772, 3600, 3964, 4157, 4346, 4432, 4546, 4576.
17. 4989, 3840, 3487, 3325, 3181, 3120, 3041, 3022.

EXERCISES XVI. Page 58.

2. 7·211, 86° 49′, 56° 19′. **3.** 36 sq. cms.
5. $c = 30$, B = 53° 08′, A = 59° 29′.
6. $c = 6·083$, B = 99° 28′, C = 43° 40′.
8. $c = 19·70$, B = 23° 58′, A = 29° 10′. **9.** 48 sq. cms.
10. $c = 44·94$, B = 32° 16′, A = 35° 07′; 336 sq. units.

EXERCISES XVII. Page 61.

1. $a=12{\cdot}65$, $B=71°34'$, $C=55°18'$; 78.
2. $a=7$, $B=81°47'$, $C=38°13'$; 17·32.
3. $c=11{\cdot}66$, $B=59°2'$, $A=98°21'$; 150.
4. $c=9{\cdot}849$, $A=66°2'$, $B=77°6'$; 72.
5. $a=16$, $B=90°$, $C=36°52'$; 96.
6. $b=26$, $A=67°23'$, $C=45°14'$; 240.
7. $a=6$, $B=70°32'$, $C=38°56°$; 11·31.
8. $c=6{\cdot}708$, $A=90°$, $B=41°48'$; 20·13.
9. $a=7$, $B=21°47'$, $C=38°13'$; 6·495.
10. $c=39$, $A=22°37'$, $B=14°15'$; 120.
11. $b=15$, $A=53°8'$, $C=14°15'$; 24.
12. $a=8{\cdot}246$, $B=43°19'$, $C=27°13'$; 11·31.
13. $a=3{\cdot}20$, $A=47°40'$, $B=80°20'$; 4·73.
14. $c=4{\cdot}55$, $B=30°20'$, $A=19°40'$; 2·30.

EXERCISES XVIII. Page 61.

2. $AC=18{\cdot}79$, $BC=19{\cdot}70$. **4.** 92·6.
5. $b=7{\cdot}660$, $a=9{\cdot}739$; Area $=29{\cdot}8$. **7.** $b=6{\cdot}84$, $a=3{\cdot}47$; Area $=5{\cdot}94$.
8. $b=6{\cdot}25$, $a=4{\cdot}91$; Area $=12{\cdot}3$.

EXERCISES XIX. Page 62.

1. 18·79. **2.** 10. **3.** $C=81°12'$, $b=9{\cdot}71$, $c=11{\cdot}94$; 38·64.
4. 36·2. **5.** $C=30°$, $a=5{\cdot}91$, $b=5{\cdot}64$.
6. $B=60°$, $a=1{\cdot}63$, $c=2{\cdot}23$. **7.** 7·17. **8.** 10. **9.** 1·61.
10. $C=60°32'$, $a=23{\cdot}64$, $c=20{\cdot}90$; Area $=82{\cdot}3$.
11. $B=38°$, $a=8{\cdot}61$, $c=15{\cdot}3$. **12.** $C=115°$, $a=3{\cdot}73$, $b=5{\cdot}67$.

EXERCISES XX. Page 63.

2. $x=8{\cdot}4$, $y=6{\cdot}6$, $A=53°8'$, $B=59°29'$, $C=67°23'$.
3. $z=11{\cdot}2$; Area $=84$. **4.** $x=2{\cdot}625$, $y=1{\cdot}375$, $A=28°57'$, $B=16°34'$, $C=104°29'$, $z=1{\cdot}453$; Area $=2{\cdot}91$.

EXERCISES XXI. Page 65.

1. $A=81°13'$, $B=61°55'$, $C=36°52'$; 210.
2. $A=41°24'$, $B=55°46'$, $C=82°50'$; 9·92.
3. $A=30°31'$, $B=59°29'$, $C=90°$; 924.
4. $A=53°8'$, $B=94°58'$, $C=31°54'$; 924.
5. $A=21°47'$, $B=38°13'$, $C=120°$; 6·50.
6. $A=38°13'$, $B=60°$, $C=81°47'$; 17·3.
7. $A=117°21'$, $B=46°24'$, $C=16°15'$; 966.
8. $A=96°44'$, $B=46°24'$, $C=36°52'$; 504.
9. $A=26°23'$, $B=36°20'$, $C=117°17'$; 5·33.
10. $A=28°4'$, $B=98°48'$, $C=53°8'$; 84.

EXERCISES XXII. Page 72.

1. Impossible. **2.** $B=90°$, $C=40°$, $c=3·214$.
3. $B=58° 20'$, $C=71° 40'$, $c=5·58$; or $B=121° 40'$, $C=8° 20'$, $c=0·85$.
4. $B=50°$, $C=80°$, $c=6·43$. **5.** $B=39° 40'$, $C=90° 20'$, $c=7·83$.
6. $A=64° 28'$, $C=70° 32'$, $a=3·828$; or $A=25° 32'$, $C=109° 28'$, $a=1·828$.
7. $A=72° 4'$, $B=47° 56'$, $a=7·690$.
8. $B=19° 28'$, $C=10° 32'$, $c=4·38$. **9.** $B=27°$, $C=90°$, $b=4·54$.
10. $A=50° 3'$, $B=94° 51'$, $b=10·40$; or $A=129° 57'$, $B=14° 57'$, $b=2·69$.

EXERCISES XXIII. Page 72.

1. $A=83° 8'$, $a=9·93$, $c=8·66$. **2.** $A=67° 23'$, $B=75° 45'$, $C=36° 52'$.
3. $a=3$, $B=48° 12'$, $C=83° 36'$. **4.** $A=25° 32'$, $B=45°$, $a=1·828$.
5. $a=2·196$, $b=2·084$, $C=138° 12'$.
6. $A=62° 44'$, $C=75° 28'$, $c=4·36$; or $A=117° 16'$, $C=20° 56'$, $c=1·61$.
7. $c=6·083$, $A=34° 43'$, $B=25° 17'$.
8. $A=39° 24'$, $B=54° 42'$, $C=85° 54'$. **9.** $B=70° 32'$, $C=90°$, $b=5·657$.
10. $B=127° 23'$, $b=10·33$, $c=6·5$.
11. $b=6·325$, $A=27° 47'$, $C=50° 47'$.
12. $A=140° 54'$, $C=24° 37'$, $a=7·57$; or $A=10° 8'$, $C=155° 23'$, $a=2·11$.

EXERCISES XXIV. Page 78.

1. $a=208$, $b=603$, $A=19°$. **2.** $b=3·11$, $c=6·67$, $C=65°$.
3. $a=204$, $c=292$, $A=35°$. **4.** $a=1290$, $c=865$, $C=42°$.
5. $b=18·1$, $c=17·8$, $A=10°$. **6.** $b=132$, $c=156$, $B=58°$.
7. $A=27° 45'$, $B=62° 15'$, $c=2·65$.
8. $B=40° 42'$, $C=49° 18'$, $a=105$.
9. $A=49° 46'$, $C=40° 14'$, $b=9740$.
10. $B=39° 1'$, $C=50° 59'$, $c=60·7$.
11. $A=46° 24'$, $C=43° 36'$, $a=463$.
12. $A=33° 16'$, $B=56° 44'$, $b=·215$.

EXERCISES XXVI. Page 80.

1. $b=55·3$, $c=86·0$, $B=40°$, $\Delta=1821$.
2. $a=147$, $c=68·5$, $C=27° 42'$, $\Delta=4468$.
3. $a=2360$, $b=2480$, $A=72° 06'$, $\Delta=9·00 \times 10^5$.
4. $b=4·99$, $c=26·5$, $C=79° 22'$, $\Delta=66·2$.
5. $b=800$, $a=536$, $A=33° 50'$, $\Delta=4·29 \times 10^5$.
6. $c=16·0$, $a=75·4$, $B=77° 44'$, $\Delta=590$.
7. $a=134$, $c=165$, $B=35° 42'$, $\Delta=6455$.
8. $b=31·6$, $c=26·7$, $C=57° 47'$, $\Delta=225$.
9. $b=1990$, $c=1920$, $A=15°$, $\Delta=4·95 \times 10^5$.
10. $a=33·8$, $b=31·8$, $C=19° 50'$, $\Delta=182$.

11. $A=50°20'$, $B=39°40'$, $c=1090$, $\Delta=2{\cdot}91\times10^5$.
12. $B=25°48'$, $C=64°12'$, $c=654$, $\Delta=1{\cdot}03\times10^5$.
13. $B=27°45'$, $C=62°15'$, $a=2650$, $\Delta=1{\cdot}45\times10^6$.
14. $A=56°29'$, $C=33°31'$, $b=213$, $\Delta=1{\cdot}045\times10^5$.
15. $A=29°54'$, $B=60°06'$, $b=49{\cdot}7$, $\Delta=710$.
16. $A=22°17'$, $C=67°43'$, $c=3310$, $\Delta=2{\cdot}25\times10^6$.
17. $A=115°34'$, $b=c=25{\cdot}3$, $\Delta=289$.
18. $B=C=68°19\frac{1}{2}'$, $a=23{\cdot}7$, $\Delta=354$.
19. $B=67°21'$, $C=45°18'$, $c=3{\cdot}56$, $\Delta=7{\cdot}58$.
20. $A=B=31°46'$, $a=b=2510$, $\Delta=2{\cdot}89\times10^6$.

EXERCISES XXVII. Page 80.

1. ·985. **2.** ·358. **3.** 4·01. **4.** ·00456. **5.** 2·025.
6. (i) 28750 ft., 25·5 secs. (ii) 41040 ft., 47·9 secs.
(iii) 44720 ft., 52·7 secs. (iv) 41040 ft., 57·1 secs.
(v) 28750 ft., 70·0 secs.
7. (i) 2616 ft. (ii) 9241 ft. (iii) 11180 ft. (iv) 13120 ft. (v) 19740 ft.
8. $\theta=39°23'$, $\phi=78°46'$. **9.** ·6549. **10.** 55°23′. **11.** 7·513.
12. 70°24′. **13.** 9°33′. **14.** 7·51. **15.** 7·51.

EXERCISES XXVIII. Page 85.

1. 109 ft. **2.** 34°7′. **3.** 32°17′. **4.** 97·5 ft. **5.** 24°27′.
6. 47·6 m., N. 33°31′ E. **7.** 5·20 m., 11·15 m. **8.** 619 m., 707 m.
9. 72·1 m., 48·65 m. **10.** S. 41°55′ W. **11.** 33·10 cms.
12. 13·81 cms. **13.** 32·6 cms., 15·73 cms.
14. 69·0 cms., 25°30′. **15.** 7·54″.
16. (i) 18·6 lbs. wt., 37°22′. (ii) 274 gms. wt., 40°3′.
17. 3·83 lbs. wt., 6·84 lbs. wt. **18.** 91·05 gms. wt., 98·65 gms. wt.
19. 336 gms. wt. **20.** 86°26′. **21.** 193 ft. **22.** 120 ft.
23. 133 yds. **24.** 30·8 m., N. 52°35′ E. **25.** 24·1 m., N. 58°55′ W.
26. 24·8 m., N. 83°54′ E. **27.** 47·9 ft., 99·45 ft. **28.** 26 ft.
29. 1990 ft. **30.** No. Angles of elevation of A, B and F in. are 5°18′, 6°09′ and 5°14′. **31.** 22·16 cms. A = 102°14′, C = 117°44′, D = 50°2′.
32. 169 sq. cms. **33.** 32·3 sq. cms. **34.** 6·12 sq. ins.
35. 15·8 cms. **36.** 147°56′, 14·73 cms.
37. PM = 9·530, QN = 9·093, AM = 7·996, AN = 12·99; PQ = 5·01 cms.
38. 3·08 ft., 3·94 ft., 2·43 ft. **39.** 3·12 ft.
40. (i) 19·3, 38°3′. (ii) 11·7, 142°22′. (iii) 625, 14°10′.
41. (40·2, 31·5) or (40·2, −12·7); (−10·6, −12·7) or (−10·6, 31·5).
42. 741 ft. **43.** 3360 ft., 7580 ft. **44.** 1·74 ft.
45. 6·82 knots. **46.** 84·7 lbs. wt., 60·4 lbs. wt.
47. 13·0 miles, N. 64°23′ E. **48.** 183 ft., N. 41°33′ E. or W.

49. S. 52° 2′ E. **51.** 117 ft. **52.** 330 ft. **53.** 6140 ft.
54. 9° 06′; 12 ft. 6·7 ins. **55.** 13° 48′. **56.** 17·37 ft.
57. 90·03 ft., 167·8 ft. **58.** 35·75 ft., 93·52 ft.
59. 29·9 ft., 38·9 ft., 16·4 ft. **60.** 4083 ft. **61.** 2774 ft.
62. 150·2 ft., 14° 58′. **63.** 29·79 ft.

EXERCISES XXIX. Page 94.

1. A = 47° 25′, B = 58° 35′, c = 48·3. **2.** B = 36° 50′, C = 32° 10′, a = 6·68.
3. A = 27° 52′, C = 110° 8′, b = 354. **4.** B = 19° 45′, C = 140° 2′, a = ·126.
5. A = 29° 11′, B = 102° 38′, c = 1530. **6.** B = 21° 13′, C = 22° 32′, a = 757.
7. C = 42°, a = 174, b = 184. **8.** A = 88°, a = 1640, c = 1600.
9. B = 35°, b = ·776, c = ·697. **10.** A = 72° 14′, a = 3580, c = 3510.
11. B = 118° 4′, a = 119, c = 305. **12.** C = 14° 26′, b = ·327, c = ·138.
13. A = 41° 49′, B = 48° 11′, C = 90°. **14.** A = 61° 25′, B = 71° 50′, C = 46° 45′.
15. A = 112° 58′, B = 27° 57′, C = 39° 5′. **16.** A = 13° 54′, B = 90°, C = 76° 6′.
17. A = 40° 12′, B = 66° 36′, C = 73° 12′.
18. A = 25° 13′, B = 104°, C = 50° 47′. **19.** B = 90°, C = 48°, c = 24·3.
20. B = 47° 35′, C = 90° 25′, c = 44·4, or B = 132° 25′, C = 5° 35′, c = 4·32.
21. B = 42°, C = 96°, c = 48·6. **22.** B = 27° 9′, C = 110° 51′, c = 66·9.
23. C = 90°, A = 35°, a = 213.
24. C = 63° 23′, A = 61° 37′, a = 365; or C = 116° 37′, A = 8° 23′, a = 60·5.
25. C = 55°, A = 70°, a = 426. **26.** C = 35° 4′, A = 89° 56′, a = 646.
27. B = 34° 50′, C = 31° 10′, c = 2·68.
28. A = 46° 12′, C = 96° 30′, c = 130; or A = 133° 48′, C = 8° 54′, c = 20·2.
29. B = 56° 38′, A = 48° 41′, a = 739. **30.** C = 16° 15′, B = 35° 28′, b = 255.
31. A = 90°, C = 35° 37′, c = 27·9. **32.** A = 78° 16′, B = 23° 28′, b = 1940.

EXERCISES XXX. Page 95.

1. 3630. **2.** 34·5. **3.** $5{\cdot}70 \times 10^6$. **4.** $5{\cdot}70 \times 10^6$.
5. $1{\cdot}23 \times 10^5$. **6.** 854. **7.** 24° 5′. **8.** $1{\cdot}07 \times 10^4$.
9. 885. **10.** $5{\cdot}30 \times 10^4$. **11.** 7·01 or 1·80.

EXERCISES XXXI. Page 98.

1. $h \cot \alpha$. **2.** (i) $c \sin B$, (ii) $b \cos B$, (iii) $a \sin B \cos B$, (iv) $\dfrac{bc}{a}$.

3. (i) $p \sin B$, (ii) $a \sin^2 B \cos B$. **4.** $2r \sin \dfrac{\theta}{2}$. **5.** $\dfrac{l}{2} \operatorname{cosec} \dfrac{\theta}{2}$.

6. $\dfrac{360°}{n}$; $2nr \sin \dfrac{180°}{n}$. **7.** $2nr \tan \dfrac{180°}{n}$. **8.** $h(\cot \alpha - \cot \beta)$.

9. $h \operatorname{cosec} \theta + a \sec \theta$. **10.** $r \cot \dfrac{\theta}{2}$, $r \operatorname{cosec} \dfrac{\theta}{2}$.

11. $OP=\sqrt{a^2+b^2+c^2}$; $\cos\alpha=\dfrac{a}{\sqrt{a^2+b^2+c^2}}$, etc.

12. (i) $p=c\sin B$, (ii) $b=\dfrac{p}{\sin C}$, (iii) $b=\dfrac{c\sin B}{\sin C}$.

13. (i) $c\cos B$, (ii) $a-c\cos B$.

14. (i) $h\cot\alpha$, (ii) $h\cot\beta$, (iii) $\dfrac{l}{\sqrt{\cot^2\beta-\cot^2\alpha}}$.

15. $\tan^{-1}\dfrac{Q\sin\alpha}{P+Q\cos\alpha}$. **16.** $\frac{1}{2}h^2(\cot\alpha+\cot\beta)$.

17. $\cos^{-1}\dfrac{r}{2l}$. **18.** $\dfrac{l\sin\alpha\sin\beta}{\sin(\beta-\alpha)}$. **19.** $\cdot 4330a^2$.

20. $\sqrt{(P\cos\alpha+Q\cos\beta+R\cos\gamma)^2+(P\sin\alpha+Q\sin\beta+R\sin\gamma)^2}$;
$\tan^{-1}\dfrac{P\sin\alpha+Q\sin\beta+R\sin\gamma}{P\cos\alpha+Q\cos\beta+R\cos\gamma}$.

21. $2\sin^{-1}\dfrac{a}{2\pi l}$. **22.** $2\sin^{-1}\dfrac{r}{d}$.

EXERCISES XL. Page 116.

1. $A=56°31'$, $B=88°42'$, $C=34°46'$. **2.** $b=3\cdot218$, $c=2\cdot717$, $A=92°12'$.
3. $B=90°$, $C=12°$, $c=6\cdot085$. **4.** $a=643$, $B=16°34'$, $C=23°18'$.
5. $C=28°46'$, $A=8°14'$, $a=2\cdot348$. **6.** $C=36°5'$, $b=918\cdot9$, $c=557\cdot2$.
7. $c=6\cdot451$, $A=44°54'$, $B=77°47'$.
8. $A=54°30'$, $B=92°21'$, $C=33°11'$.
9. $A=79°48'$, $B=59°12'$, $a=89\cdot60$, or $A=18°12'$, $B=120°48'$, $a=28\cdot43$.
10. $b=1\cdot027$, $A=50°33'$, $C=61°17'$.
11. $A=39°26'$, $B=97°8'$, $C=43°27'$. **12.** $B=39°43'$, $a=406\cdot2$, $c=191\cdot2$.
13. $a=546\cdot3$, $B=90°$, $C=51°19'$. **14.** $C=49°34'$, $A=7°26'$, $a=135$.
15. $C=35°45'$, $b=2\cdot162$, $c=2\cdot836$. **16.** $A=35°38'$, $B=54°21'$, $C=90°$.
17. $C=55°28'$, $B=87°9'$, $b=112\cdot6$, or $C=124°32'$, $B=18°5'$, $b=34\cdot97$.
18. $A=44°8'$, $B=107°42'$, $C=28°10'$.
19. $B=120°26'$, $a=41\cdot94$, $c=45\cdot86$.
20. $c=799\cdot1$, $A=14°29'$, $B=42°15'$.

EXERCISES XLI. Page 118.

1. 18·33 m., N. 70° 54′ E. **2.** 197 yds.
3. N. 30° 31′ E., N. 36° 52′ W. **4.** 19·08 m., N. 57° E.
5. N. 15° 59′ E. **6.** 9·17 m., S. 13° 06′ W. **7.** 3·50 m., 7·34 m.
8. 34° 14′, 48° 36′, 3·306 ft. **9.** 4·59 ft., 7·25 ft. **10.** 88·4 ft.
11. 58° 31′ 44° 25′. **12.** 163·8 ft. **13.** 4·472″. **14.** 6·782 cms.
15. 50° 59′, 129° 01′, 92° 44′, 87° 16′. **16.** 110·35 ft.
17. 52° 37′, 80° 25′, 110° 29′. **18.** ·824, ·372, ·172, ·628 ft.

19. 4·359 lbs. wt., at 23° 25′ with greater force.
8·544 ,, ,, 16° 19′ ,, ,,
7 ,, ,, 38° 13′ ,, ,,
20. 62° 43′, 153° 37′, 143° 40′. **21.** 1·513, 9·913 cms. **22.** 3·02 cms.
23. 91° 21′, 59° 29′, 89° 10′. **24.** 33 55 mins. **25.** 81° 47′.
26. 111 ft. **27.** 119·7 ft. **28.** 90·14 ft., N. 73° 54′ E.
29. 140 ft., N. 6° 47′ E. **30.** 269·5 ft.

EXERCISES XLII. Page 125.

1. 9·43 miles, 10·73 miles. **2.** 6·53 miles, N. 10° E.
3. 23·85 miles, S. 53° 11′ W. **4.** S. 86° 37′ E., N. 40° 1′ E.
5. 20·1 miles. **6.** 631 yds. **7.** 13·85 knots, N. 77° 40′ E.
8. 8·83 ft., 81° 32′. **9.** 72·7 ft., 1794 sq. ft.
10. 53°, 80° 58′, 111° 7′. **11.** 10·36″, 2·17″, 11·22″ from mean position.
12. 7·25 lbs. wt. at an angle of 19° 40′ with greater force.
422 gms. wt. ,, ,, 48° 48′ ,, ,, ,,
13. 137° 32′, 118° 50′, 103° 38′. **14.** 1387 yds. **15.** 75·5 miles.
16. 9·30 miles, N. 1° 39′ E. **17.** 1502 yds. **18.** 274 ft.
19. 6710 ft. **20.** 19·24 mins. **21.** 75·2 ft. **22.** 454 ft.
23 1027 yds. **24.** 12 h. 36 m., p.m. **25.** 45·9 yds., 80·9 yds.
26. 1027 ft. **27.** 114 yds. **28.** 109° 06′. **29.** 7·66 m., S. 80° E
30. 1317 yds. **31.** 8·43 m., S. 35° 35′ E. **32.** 7·98 m., S. 88° 15′ W.
33. 23·19 knots. **34.** 17·7 mins. **35.** 9·81 m., S. 81° 40′ E.
36. 38° 23′, 53° 24′, 15° 01′, 30° 02′, 2·27 ins.
37. 3·09 ins., 1·865 ins., 76° 30′, 171° 30′, 58° 03′, 1·69 ins.
38. 8·12 m. **39.** 10·8 knots. **40.** 63·82″, 56·85″. **41.** 66·3 ft.
42. 206·3 ft. **43.** S. 48° 20′ W. **44.** N. 68° 27′ W.
45. 1476 yds. **46.** 94·3 yds., 19° 01′, 9° 13′. **47.** 84·3 ft.
48. 449·0 yds., 448·6 yds. **49.** 1963 ft.
50. 37° 52′, 60° 15′, 81° 52′.

EXERCISES XLIII. Page 132.

1. 24·392 and 24·408. **2.** 685·4 and 686·4.
3. 72·95 and 73·05. **4.** ·004556 and ·004564.
5. 3·127 and 3·129; 3·138 and 3·142; 3·135 and 3·145.
6. 3·089 and 3·091; 3·126 and 3·130; 3·135 and 3·145.
8. 42·07 and 42·11. **9.** 47° 07′ and 47° 11′.
10. 70·63 ft. and 74·98 ft.; 212·62 ft. and 227·33 ft.
11. 55·23 ft. and 56·69 ft. **12.** 173·16 ft. and 184·30 ft.
13. 3° 25′ and 3° 45′. **14.** 78·45 yds. and 95·85 yds
15. 33·19 yds. and 35·25 yds.
16. 134·4 ft. and 137·7 ft.; 19·53 ft. and 27·65 ft.
17. 166·5 ft. and 173·1 ft. **18.** 11·22 m. and 12·23 m.
19. 69·4 m. and 82·2 m.

MISCELLANEOUS EXERCISES. Page 134.

A. **1.** ·71, ·98. **2.** ·088. **4.** 7·15 cms., 55·68 cms.
5. 3844 yds., N. 51° 28′ E.
B. **1.** $C=33°$, $b=10·06$, $c=6·54$. **4.** 44° 03′. **5.** 110·5 sq. ft.
C. **1.** ·96. **2.** $b=1·77$, $c=5·30$, $B=19°28′$, $A=70°32′$. **4.** 11·15″.
5. ·08″, ·875″.
D. **2.** $a=b=10·02$, $B=37°$, $C=106°$. **3.** $\frac{8}{17}$. **4.** 16·79 cms.
5. 1600 ft.
E. **2.** 1·36″, 3·36″. **4.** 3·49″. **5.** 3·63 cms.
F. **2.** $D=56°19′$, $A=123°41′$, $AD=3·606″$.
G. **1.** $B=34°35′$, $C=55°25′$, $a=23·4$. **2.** 36° 52′. **3.** ·184.
4. 19·73. **5.** 13° 19′.
H. **1.** $\frac{1}{5}$, 5. **2.** 57° 15′. **3.** 28·2 lbs. wt.; $2P\cos\frac{a}{2}$.
5. 38·2 ft., 54·5 ft., 45·5 ft.
I. **1.** $AD=2·414″$, $CD=2·614″$. **4.** 10° 29′, 46° 39′.
5. 1·732 m., 1 m.
J. **1.** (i) 903 m., (ii) 64° 20′. **2.** 33° 41′, 79° 47′.
3. $A=B=75°01′$, $C=29°58′$. **4.** 1·63″. **5.** 2850 ft.
K. **1.** $A=48°12′=B$. **2.** 4·415 cms., 2·347 cms., 2·067 cms., 6·762 cms. **3.** 9·703 cms., 7·581 cms., 12·419 cms. **4.** 2·115″.
5. 82·1″.
L. **2.** $AD=3·732″$, $CD=3·864″$. **3.** $B=65°42′$, $C=24°18′$, $b=14·58$, $c=6·58$. **4.** 7·86 cms. **5.** 116 ft.
M. **1.** 36° 52′. **2.** 47° 42′. **3.** 1·24″. **4.** 3·63 : 1.
5. 30° 26′.
N. **1.** 28° 51′. **2.** ·925. **3.** 120°. **4.** 5·292″.
5. (i) $\frac{\sin P}{\sin 60°}$, (ii) $\frac{\sin R}{\sin 30°}$, (iii) $\frac{\tan R}{\tan 30°}$, (iv) S. 40° 54′ W.
O. **1.** ·125. **2.** 7·211″. **3.** 72° 32′. **4.** 2646 ft.
5. $a=b-c$, $a=b+c$.
P. **1.** 14° 29′ E., 48° N. **3.** 1·082″. **4.** 5·236.
5. 174 yds., S. 22° E.
Q. **1.** 41·91. **2.** 82·45 sq. cms. **3.** 9·405″, 15·22″.
4. 29° 45′. **5.** 22·34 ft.
R. **2.** 206 secs. **3.** 8·25 sq. ins. **5.** 10·40 ft.
S. **1.** 2·08″. **2.** 5·491″, 7·396″, 8·412″. **3.** 18·76″ or 8·96″; 48° 36′. **4.** 3″ from A. **5.** 59·2 ft., 39·6 ft.
T. **1.** (i) Impossible, (ii) 31° 10′, (iii) 14° 29′. **2.** 17·98 cms.
4. 38·7 cms., 27·8 cms., 25·0 cms. **5.** 46·2 ft., 100 ft.

U. **1.** $13°\,02'$. **2.** $83°\,54'$, $126°\,41'$, $149°\,25'$.

3. $\frac{d}{2}(1-\cos\theta)$, $\frac{d}{2}(1+\cos\theta)$. **4.** $29°\,52'$.

5. 5395 yds., S. $89°\,10'$ W.

V. **1.** $\theta=45°$, distance of PQ from AB $=\cdot 7071a$.

3. 323 gms. wt., 192 gms. wt., $\frac{R\sin\beta}{\sin(\alpha+\beta)}$, $\frac{R\sin\alpha}{\sin(\alpha+\beta)}$.

4. 3·81 cms., 14·19 cms. **5.** N. $53°\,48'$ E.

W. **1.** 3·37. **2.** 5 ($x=36°\,52'$); 0 ($x=126°\,52'$).

4. 115·5 cms., 103·8 cms. **5.** 86·4 ft.

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